一、微分流形与張量场

1、張量

· 标量场: $T'(x')=T(x)$

· 矢量场: $\begin{cases}\text{逆变矢量场}: & T'^{\mu}(x')=\frac{\partial x'^{\mu}}{\partial x^{\nu}}T^{\nu}(x)\\ \text{协变矢量场}: & T'_{\mu}(x')=\frac{\partial x^{\nu}}{\partial x'^{\mu}}T_{\nu}(x)\end{cases}$

坐标微分是一个逆变矢量场 $dx'^{\mu}=\frac{\partial x'^{\mu}}{\partial x^{\nu}}dx^{\nu}$

· 二阶张量场

逆变张量场: $T'^{\mu\nu}(x')=\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}T^{\rho\sigma}(x)$

协变张量场: $T'_{\mu\nu}(x')=\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}T_{\rho\sigma}(x)$

混合张量场: $T'^{\mu}{}_{\nu}(x')=\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}T^{\rho}{}_{\sigma}(x)$

$[A^{\mu},B^{\nu}]^{\lambda}=[A,B]=[A,B]^{\lambda}=A^{\mu}\partial_{\mu}B^{\lambda}-B^{\mu}\partial_{\mu}A^{\lambda}$ 仍为逆变矢量场

proof:

$$[A'^{\mu},B'^{\nu}]^{\lambda}=A'^{\mu}\partial'_{\mu}B'^{\lambda}-B'^{\mu}\partial'_{\mu}A'^{\lambda}=\frac{\partial x'^{\mu}}{\partial x^{\rho}}A^{\rho}\frac{\partial x^{\sigma}}{\partial x'^{\mu}}\partial_{\sigma}\left(\frac{\partial x'^{\lambda}}{\partial x^{\tau}}B^{\tau}\right)-\frac{\partial x'^{\mu}}{\partial x^{\rho}}B^{\rho}\frac{\partial x^{\sigma}}{\partial x'^{\mu}}\partial_{\sigma}\left(\frac{\partial x'^{\lambda}}{\partial x^{\tau}}A^{\tau}\right)$$

$$=A^{\rho}\delta_{\rho}^{\sigma}\partial_{\sigma}\left(\frac{\partial x'^{\lambda}}{\partial x^{\tau}}B^{\tau}\right)-B^{\rho}\delta_{\rho}^{\sigma}\partial_{\sigma}\left(\frac{\partial x'^{\lambda}}{\partial x^{\tau}}A^{\tau}\right)$$

$$=\frac{\partial x'^{\lambda}}{\partial x^{\tau}}\left[A^{\rho}\partial_{\rho}B^{\tau}-B^{\rho}\partial_{\rho}A^{\tau}\right]+A^{\rho}(\partial_{\rho}\partial_{\tau}x'^{\lambda})B^{\tau}-B^{\rho}(\partial_{\rho}\partial_{\tau}x'^{\lambda})A^{\tau}$$

$$=\frac{\partial x'^{\lambda}}{\partial x^{\tau}}[A^{\rho},B^{\sigma}]^{\tau}+2\partial_{(\rho}\partial_{\tau)}x'^{\lambda}A^{[\rho}B^{\tau]}$$

$$=\frac{\partial x'^{\lambda}}{\partial x^{\tau}}[A^{\rho},B^{\sigma}]^{\tau}.$$

$\delta^{\mu}{}_{\nu}=\begin{cases}1, & \mu=\nu\\ 0, & \mu\neq\nu\end{cases}$ 不依赖于坐标系的选取.

$\delta'^{\mu}{}_{\nu}=\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\delta^{\rho}{}_{\sigma}=\frac{\partial x'^{\mu}}{\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}=\frac{\partial x'^{\mu}}{\partial x'^{\nu}}=\delta^{\mu}{}_{\nu}$.

对称張量: $T^{(\mu\nu)}=\frac{1}{2}(T^{\mu\nu}+T^{\nu\mu})$

反称張量: $T^{[\mu\nu]}=\frac{1}{2}[T^{\mu\nu}-T^{\nu\mu}]$

· 对称、反称指标缩并为 0

· n维流形m阶反称张量有 C_n^m 个独立分量.

2、外微分与微分形式

微分形式: 完全反对称的对偶張量场或者带基的 (0,p)型全反对称張量场。

一个(0,p)型全反对称张量场(带基) $T_{[a_1\cdots a_p]}$ 称为 p形式:

· 标量场: 0形式: f

· 对偶矢量场: 1形式: $\alpha=\alpha_{\mu}dx^{\mu}$

• 2形式：$\underline{\alpha_2} = \frac{1}{2}\alpha_{\mu\nu} dx^\mu \wedge dx^\nu = \sum_{\mu<\nu} \alpha_{\mu\nu} dx^\mu \wedge dx^\nu$

2形式空间基矢 $dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$，维数为 C_n^2、

$n=2$. $\underline{\alpha_2} = \alpha_{12} dx^1 \wedge dx^2$

$n=3$ $\underline{\alpha_2} = \alpha_{12} dx^1 \wedge dx^2 + \alpha_{13} dx^1 \wedge dx^3 + \alpha_{23} dx^2 \wedge dx^3$

外积

p形式 $\underline{\alpha_p} = \alpha_{[\mu_1 \cdots \mu_p]} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ 与 q形式 $\underline{\beta_q} = \beta_{[\mu_1 \cdots \mu_q]} dx^{\mu_1} \wedge \cdots dx^{\mu_q}$ 的外积为

$$\underline{\alpha_p} \wedge \underline{\beta_q} = \frac{(p+q)!}{p!\,q!} \alpha_{[a_1 a_2 \cdots a_p} \beta_{b_1 b_2 \cdots b_q]} = (-1)^{pq} \underline{\beta_q} \wedge \underline{\alpha_p}$$

结合律：$(\underline{\alpha} \wedge \underline{\beta}) \wedge \underline{\gamma} = \underline{\alpha} \wedge (\underline{\beta} \wedge \underline{\gamma})$

分配律：$(a\alpha + b\beta) \wedge \gamma = a(\alpha \wedge \gamma) + b(\beta \wedge \gamma)$

$\underline{\alpha} \wedge (a\beta + b\gamma) = a(\alpha \wedge \beta) + b(\alpha \wedge \gamma)$

斜交换律：$\underline{\alpha_p} \wedge \underline{\beta_q} = (-1)^{pq} \underline{\beta_q} \wedge \underline{\alpha_p}$

外微分：$d: \ \Lambda^p \to \Lambda^{p+1}$

a. $df = \partial_\mu f \, dx^\mu$

b. $d(df) = d^2 f = 0 \Rightarrow d(d\underline{\alpha}) = d^2 \underline{\alpha} = 0$

c. $d(a\underline{\alpha} + b\underline{\beta}) = a\, d\underline{\alpha} + b\, d\underline{\beta}$

d. $d(\underline{\alpha_p} \wedge \underline{\beta_q}) = (d\underline{\alpha_p}) \wedge \underline{\beta_q} + (-1)^p \underline{\alpha_p} \wedge (d\underline{\beta_q})$

$\Rightarrow d(f\underline{\alpha_p}) = df \wedge \underline{\alpha_p} + f\, d\underline{\alpha_p}$

e. $d\underline{\alpha_p} = \frac{1}{p!(p+1)!} \delta^{\nu\mu_1 \cdots \mu_p}{}_{k_1 \cdots k_{p+1}} (\partial_\nu \alpha_{\mu_1 \cdots \mu_p}) dx^{k_1} \wedge dx^{k_2} \wedge \cdots \wedge dx^{k_{p+1}}$

对1形式有

$d\underline{\alpha} = \frac{1}{2} \delta^{\nu\mu}{}_{k_1 k_2} \partial_\nu \alpha_\mu \, dx^{k_1} \wedge dx^{k_2} =$

$d\underline{\alpha} = 0$ 称 $\underline{\alpha}$ 为闭形式

$\underline{\alpha_p} = d\underline{\beta_{p-1}}$ 称 $\underline{\alpha_p}$ 为恰当形式 $\Rightarrow$ 恰当形式必定为闭形式.

3. 度规

$g_{\mu\nu}$：非退化(行列式不为0)、对称、2阶协变张量场.

① 定义流形两相邻点间隔：$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

② 升降张量指标

③ 定义两矢量场内积 $g_{\mu\nu} A^\mu B^\nu$

对角化后的所有对角元

• 恒正：正定度规 → 黎曼空间

。有正有负：不定度规 $\longrightarrow$ 伪黎曼空间

$\xrightarrow[\text{全正}]{\text{对角元-负其余}}$ 黎曼空间

4. 曲线

曲线：$I \subset \mathbb{R}$，$C: I \to M$ 称为 M 上的一条 C^r 类曲线.

切矢：设曲线 $\psi_\alpha \circ C: I \in \mathbb{R} \to \mathbb{R}^n$ 坐标为 $x^\mu(\sigma)$，则该曲线切矢为：$t^\mu = \frac{dx^\mu}{d\sigma}$

对坐标曲线 x^μ 切矢：$\left(\frac{\partial}{\partial x^\mu}\right)^a$

切矢量场在重参数变换下的变换关系：$\frac{dx'^\mu}{d\sigma'}\frac{d\sigma'}{d\sigma} = \frac{dx^\mu}{d\sigma}$

积分曲线：对给定矢量场 t^μ，曲线 $c(\sigma)$ 上每点切矢量 等于该点的 t^μ.

重参数化：$C: I \to M$ 与 $C': I' \to M$ 的像相同，可视为同一曲线.

曲线任一点 p 的切矢可分为以下三类：

$$
g_{\mu\nu} t^\mu t^\nu \big|_p = \begin{cases} <0 & \text{类时} \xrightarrow[\forall p \in C]{\text{类时曲线}} \text{质点运动（世界线）} \\ >0 & \text{类空} \\ =0 & \text{类光} \xrightarrow[\forall p \in C]{\text{类光曲线}} \text{零曲线} \end{cases}
$$

5 测地线

测地线方程：$t^\lambda \nabla_\lambda t^\mu = 0$ 或者 $\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0$ 或 $\frac{Dt^\mu}{d\sigma} = 0$

推导类空极值曲线满足测地线方程.

$$dl^2 = g_{\mu\nu} dx^\mu dx^\nu \Rightarrow dl = (g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}}$$

$$l = \int_{t_1}^{t_2} (g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}} = \int_{t_1}^{t_2} \sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\, dt$$

由于 $\begin{cases} \delta g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\delta x^\sigma \\ \delta\left(\frac{dx^\mu}{dt}\right) = \frac{d}{dt}(\delta x^\mu) \end{cases}$

有
$$\delta l = \frac{1}{2}\int_{t_1}^{t_2} \left(g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right)^{-\frac{1}{2}}\left[(\delta g_{\mu\nu})\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + g_{\mu\nu}\delta\left(\frac{dx^\mu}{dt}\right)\frac{dx^\nu}{dt} + g_{\mu\nu}\frac{dx^\mu}{dt}\delta\left(\frac{dx^\nu}{dt}\right)\right] dt$$

$$= \frac{1}{2}\int_{t_1}^{t_2} \left(g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right)^{-\frac{1}{2}}\left[\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\delta x^\sigma\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + g_{\mu\nu}\frac{d}{dt}(\delta x^\mu)\frac{dx^\nu}{dt} + g_{\nu\mu}\frac{dx^\nu}{dt}\delta\left(\frac{dx^\mu}{dt}\right)\right] dt$$

$$= \frac{1}{2}\int_{t_1}^{t_2} \left(g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right)^{-\frac{1}{2}}\left[\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\delta x^\sigma + 2g_{\mu\nu}\frac{dx^\nu}{dt}\frac{d}{dt}(\delta x^\mu)\right] dt$$

无论曲线原参数取如何，总可选新参数 $t = t(\tilde{t})$ 使曲线上每点切矢长度归一，即 $g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 1$

从而
$$\delta l = \int_{t_1}^{t_2}\left[\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\delta x^\sigma + g_{\mu\nu}\frac{dx^\nu}{dt}\frac{d}{dt}(\delta x^\mu)\right] dt$$

$$= \int_{t_1}^{t_2}\left[\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\delta x^\sigma + \frac{d}{dt}\left(g_{\mu\nu}\frac{dx^\nu}{dt}\delta x^\mu\right) - \frac{d}{dt}\left(g_{\mu\nu}\frac{dx^\nu}{dt}\right)\delta x^\mu\right] dt$$

$$= \int_{t_1}^{t_2}\left[\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\delta x^\sigma - \frac{d}{dt}\left(g_{\mu\nu}\frac{dx^\nu}{dt}\right)\delta x^\mu\right] + g_{\mu\nu}\frac{dx^\nu}{dt}\delta x^\mu\Big|_{t_1}^{t_2}$$

$$= \int_{t_1}^{t_2}\left[\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} - \frac{d}{dt}\left(g_{\sigma\nu}\frac{dx^\nu}{dt}\right)\right](\delta x^\sigma)\, dt = 0$$

即 $\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} - \frac{d}{dt}g_{\sigma\nu}\frac{dx^\nu}{dt} - g_{\sigma\nu}\frac{d^2x^\nu}{dt^2} = 0$

$\Rightarrow \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} - \frac{\partial g_{\sigma\nu}}{\partial x^\mu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} - g_{\sigma\nu}\frac{d^2x^\nu}{dt^2} = 0$

以 $g^{\sigma\rho}$ 缩并上式有：

$$g^{\sigma\rho}\left(\frac{1}{2}g_{\mu\nu,\sigma} - g_{\sigma\nu,\mu}\right)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} - \frac{d^2x^\rho}{dt^2} = 0$$

$$\Rightarrow \frac{1}{2} g^{\sigma\rho}(g_{\mu\nu,\sigma} - g_{\sigma\nu,\mu} - g_{\sigma\mu,\nu}) \frac{dx^\mu}{dt}\frac{dx^\nu}{dt} - \frac{d^2x^\rho}{dt^2} = 0$$

即 $\frac{d^2x^\rho}{dt^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0$，$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})$.

6. 黎曼正则坐标与等效原理

在流形某点 p 的邻域，总存在着黎曼正则坐标系，使得度规及其导数满足

$$\begin{cases} g_{\mu\nu}|_p = \eta_{\mu\nu} \\ \partial_\sigma g_{\mu\nu}|_p = 0 \\ \partial_\sigma \partial_\rho g_{\mu\nu}|_p = 0 \end{cases}$$

等效原理的表述：

弱等效原理 (WEP)：惯性质量与引力质量相等，即 $m_i = m_g$.（或引力场与惯性场的力学效应是局域不可分辨的）

爱因斯坦等效原理 (EEP)：在足够小区域，物理学定律应与狭义相对论中的一致，即无法通过局域实验探测出引力场的存在.

强等效原理 (SEP)：任何局域测试实验结果与自由下落装置的速度无关，与实验在何时何处完成的无关.

数学表述：在黎曼时空中任一点 p 都存在一个局域坐标系，其中 p 点克氏符的所有分量为 0.

7. 流形上的映射

参考梁书上册第4章

8. Lie 导数

定义：张量场 $T^{a_1a_2\cdots a_k}{}_{b_1b_2\cdots b_l}$ 沿矢量场 v^a 的 Lie 导数为：

$$\mathcal{L}_v T^{a_1a_2\cdots a_k}{}_{b_1b_2\cdots b_k} = \lim_{t\to 0} \frac{1}{t}(\phi_t^* T^{a_1\cdots a_k}{}_{b_1\cdots b_l} - T^{a_1\cdots a_k}{}_{b_1\cdots b_l})$$

a. $\mathcal{L}_v f = v(f) = v^\mu \partial_\mu f$，$\forall f$

在适配坐标系下有

b. $\mathcal{L}_v u^a = [v,u]^a = v^b D_b u^a - u^b D_b v^a$，$\forall u^a, v^a \in \mathcal{F}(1,0)$.

c. $\mathcal{L}_v w_a = v^b D_b w_a + w_b D_a v^b$，$\forall v^a \in \mathcal{F}(1,0)$，$w^a \in \mathcal{F}(0,1)$.

proof: 对 b. 在适配坐标系下有

$$[v,u]^\mu = v^\nu \partial_\nu u^\mu - u^\nu \partial_\nu v^\mu = v^\nu \partial_\nu u^\mu - u^\nu \partial_\nu (\frac{\partial}{\partial x^1})^\mu = v^\nu \partial_\nu u^\mu - u^\nu \partial_\nu \delta^\mu{}_1 = v^\nu \partial_\nu u^\mu$$

$$= v(u^\mu) = \frac{\partial}{\partial x^1} u^\mu = (\mathcal{L}_v u)^\mu$$

对 c. 由于 $\mathcal{L}_v(w_a u^a) = (\mathcal{L}_v w_a) u^a + w_a (\mathcal{L}_v u^a) = (\mathcal{L}_v w_a) u^a + w_a (v^b D_b u^a - u^b D_b v^a)$

$$= (\mathcal{L}_v w_a) u^a + \underline{w_a v^b D_b u^a} - w_b u^a D_a v^b$$

$$\mathcal{L}_v(w_a u^a) = v(w_a u^a) = v^b D_b (w_a u^a) = v^b (D_b w_a) u^a + \underline{v^b w_a D_b u^a}$$

从而 $(\mathcal{L}_v w_a) u^a = v^b (D_b w_a) u^a + w_b u^a D_a v^b$

即 $\mathcal{L}_v w_a = v^b D_b w_a + w_b D_a v^b$

d. Cartan 公式：

$\mathcal{L}_v = d \circ i_v + i_v \circ d$，$i_v$ 为内积或者收缩运算，代表矢量场与形式之间的内积.

$$\left.\begin{aligned} d\circ \mathcal{L}_v &= d\circ d \circ i_v + d\circ i_v \circ d = d\circ i_v \circ d \\ \mathcal{L}_v \circ d &= d\circ i_v \circ d + i_v \circ d \circ d = d \circ i_v \circ d \end{aligned}\right\} \Rightarrow d\circ \mathcal{L}_v = \mathcal{L}_v \circ d$$

9. Hodge 对偶

定义 Hodge 星算符 $*\ \wedge^p \to \wedge^{n-p}$ 使得 $\forall A_{\nu_1\cdots\nu_p} \in \wedge^p$ 满足 $(*A)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\varepsilon^{\nu_1\cdots\nu_p}{}_{\mu_1\cdots\mu_{n-p}} A_{\nu_1\cdots\nu_p}$

$(*f)_{\mu_1\cdots\mu_n} = \frac{1}{0!} f \varepsilon_{\mu_1\cdots\mu_n} = f\varepsilon_{\mu_1\cdots\mu_n}$

$*(*f) = \frac{1}{n!}\varepsilon^{\mu_1\cdots\mu_n} f \varepsilon_{\mu_1\cdots\mu_n} = \frac{1}{n!}(-1)^s n! f = (-1)^s f$. S为度规在正交归一基底中(-1)的个数.

对二维欧氏空间, $(-1)^s = (-1)^0 = 1$

$(*dx)_\mu = \frac{1}{1!}\varepsilon^\nu{}_\mu (dx)_\nu = \varepsilon_{\nu\mu}(dx)^\nu = dy$

$*dy = -dx$

例: 二维欧氏空间: $ds^2 = dx^2 + dy^2$, $f(x,y)$为其上的标量场. 计算$d{*}df$.

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$\Rightarrow *(df) = \frac{\partial f}{\partial x}*(dx) + \frac{\partial f}{\partial y}*(dy) = \frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx$

$d(*(dy)) = d(\frac{\partial f}{\partial x})\wedge dy + \frac{\partial f}{\partial x}d(dy) - d(\frac{\partial f}{\partial y})\wedge dx - \frac{\partial f}{\partial y}d(dx)$

$= (\frac{\partial^2 f}{\partial x^2}dx + \frac{\partial^2 f}{\partial x\partial y}dy)\wedge dy - (\frac{\partial^2 f}{\partial y\partial x}dx + \frac{\partial^2 f}{\partial y^2}dy)\wedge dx$

$= \frac{\partial^2 f}{\partial x^2}dx\wedge dy - \frac{\partial^2 f}{\partial y^2}dy\wedge dx$

$= (\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2})\, dx\wedge dy$

10. 协变导数与联络

$\Gamma'^{\tau}{}_{\lambda k} = \frac{\partial x'^\tau}{\partial x^\rho}\frac{\partial x^\mu}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^k}\Gamma^\rho{}_{\mu\sigma} + \frac{\partial x'^\tau}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\lambda\partial x'^k} \Rightarrow D_\lambda A^\nu = \partial_\lambda A^\nu + \Gamma^\nu{}_{\mu\lambda}A^\mu$

仿射联络系数

$D_\lambda B_\mu = \partial_\lambda B_\mu - \Gamma^\sigma{}_{\lambda\mu}B_\sigma$

Proof: $D_\lambda(A^\mu B_\mu) = (D_\lambda A^\mu)B_\mu + A^\mu D_\lambda B_\mu = (\partial_\lambda A^\mu + \Gamma^\mu{}_{\nu\lambda}A^\nu)B_\mu + A^\mu D_\lambda B_\mu$

$D_\lambda(A^\mu B_\mu) = \partial_\lambda(A^\mu B_\mu) = (\partial_\lambda A^\mu)B_\mu + A^\mu\partial_\lambda B_\mu$

$\Rightarrow \Gamma^\mu{}_{\nu\lambda}A^\nu B_\mu + A^\mu D_\lambda B_\mu = A^\mu\partial_\lambda B_\mu$, $\forall A^\mu$

$\Gamma^\sigma{}_{\mu\lambda}A^\mu B_\sigma + A^\mu D_\lambda B_\mu = A^\mu\partial_\lambda B_\mu$.

即 $D_\lambda B_\mu = \partial_\lambda B_\mu - \Gamma^\sigma{}_{\mu\lambda}B_\sigma$

证明 $D_\lambda B_\mu$ 为 $(0,2)$ 型张量

$D'_\lambda B'_\mu = \partial'_\lambda B'_\mu - \Gamma'^\nu{}_{\lambda\mu}B'_\nu = \frac{\partial x^\rho}{\partial x'^\lambda}\partial_\rho(\frac{\partial x^\sigma}{\partial x'^\mu}B_\sigma) - (\frac{\partial x'^\nu}{\partial x^\tau}\frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\mu}\Gamma^\tau{}_{\alpha\beta} + \frac{\partial x'^\nu}{\partial x^\tau}\frac{\partial^2 x^\tau}{\partial x'^\lambda\partial x'^\mu})(\frac{\partial x^\gamma}{\partial x'^\nu}B_\gamma)$

$= \frac{\partial x^\rho}{\partial x'^\lambda}(\frac{\partial^2 x^\sigma}{\partial x'^\mu\partial x^\rho}B_\sigma + \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial B_\sigma}{\partial x^\rho}) - (\frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\mu}\Gamma^\gamma{}_{\alpha\beta} + \frac{\partial^2 x^\gamma}{\partial x'^\lambda\partial x'^\mu})B_\gamma$

$= \frac{\partial^2 x^\sigma}{\partial x'^\mu\partial x'^\lambda}B_\sigma + \frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial B_\sigma}{\partial x^\rho} - \frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\mu}\Gamma^\gamma{}_{\alpha\beta}B_\gamma - \frac{\partial^2 x^\gamma}{\partial x'^\lambda\partial x'^\mu}B_\gamma$

$= \frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial B_\sigma}{\partial x^\rho} - \frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\mu}\Gamma^\gamma{}_{\alpha\beta}B_\gamma$

$= \frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial B_\beta}{\partial x^\alpha} - \frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\mu}\Gamma^\gamma{}_{\alpha\beta}B_\gamma$

$= \frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\mu}[B_{\beta,\alpha} - \Gamma^\gamma{}_{\alpha\beta}B_\gamma]$

$= \frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\mu}D_\alpha B_\beta$

联络系数各指标间无对称性，可分解为2个部分.

$$\Gamma^{\mu}_{\nu\lambda}=\Gamma^{\mu}_{(\nu\lambda)}+\Gamma^{\mu}_{[\nu\lambda]}=\frac{1}{2}(\Gamma^{\mu}_{\nu\lambda}+\Gamma^{\mu}_{\lambda\nu})+\frac{1}{2}(\Gamma^{\mu}_{\nu\lambda}-\Gamma^{\mu}_{\lambda\nu})$$

对称　　　　反对称：挠率张量

一般联络系数有n^3个独立分量

$n=4$　$4^3=64$个 $=40+24$ 独立分量

对称　反称

黎曼几何基本定理：黎曼时空流形(M,g)上存在唯一的与度规g适配的无挠协变导数算子.

克氏符 $\Gamma^{\sigma}_{\mu\nu}=\frac{1}{2}g^{\sigma\rho}(g_{\rho\mu,\nu}+g_{\rho\nu,\mu}-g_{\mu\nu,\rho})$

对闵氏空间：$\Gamma^{\lambda}_{\mu\nu}=0$

对 $ds^2=dr^2+r^2d\theta^2$

$$\begin{cases}g_{rr}=1\\ g_{\theta\theta}=r^2\end{cases}\Rightarrow\begin{cases}g^{rr}=1\rightarrow\Gamma^{r}_{\mu\nu}=\frac{1}{2}(g_{r\mu,\nu}+g_{r\nu,\mu}-g_{\mu\nu,r})\\ g^{\theta\theta}=r^{-2}\rightarrow\Gamma^{\theta}_{\mu\nu}=\frac{1}{2}r^{-2}(g_{\theta\mu,\nu}+g_{\theta\nu,\mu}-g_{\mu\nu,\theta})\end{cases}$$

只有 $\Gamma^{r}_{\theta\theta}=\frac{1}{2}(-g_{\theta\theta,r})=-r$

$\Gamma^{\theta}_{\theta r}=\Gamma^{\theta}_{r\theta}=\frac{1}{2}r^{-2}(g_{\theta\theta,r}+g_{\theta r,\theta}-g_{\theta r,\theta})=\frac{1}{r}$

两个非零联络系数.

3维欧氏空间中的2维单位球面

$ds^2=d\theta^2+\sin^2\theta d\varphi^2$

$$\begin{cases}g_{\theta\theta}=1\\ g_{\varphi\varphi}=\sin^2\theta\end{cases}\Rightarrow\begin{cases}g^{\theta\theta}=1\Rightarrow\Gamma^{\theta}_{\mu\nu}=\frac{1}{2}(g_{\theta\mu,\nu}+g_{\theta\nu,\mu}-g_{\mu\nu,\theta})\\ g^{\varphi\varphi}=\sin^{-2}\theta\Rightarrow\Gamma^{\varphi}_{\mu\nu}=\frac{1}{2}\sin^{-2}\theta(g_{\varphi\mu,\nu}+g_{\varphi\nu,\mu}-g_{\mu\nu,\varphi})\end{cases}$$

只有 $\Gamma^{\theta}_{\varphi\varphi}=-\frac{1}{2}g_{\varphi\varphi,\theta}=-\sin\theta\cos\theta$

$\Gamma^{\varphi}_{\theta\varphi}=\Gamma^{\varphi}_{\varphi\theta}=\frac{1}{2}\sin^{-2}\theta(g_{\varphi\varphi,\theta}+g_{\varphi\theta,\varphi}-g_{\varphi\theta,\varphi})=\cot\theta$

11. 张量场沿曲线的平行移动

逆变矢量A^{μ}满足 $t^{\lambda}D_{\lambda}A^{\nu}=0$. 称$A^{\mu}$沿$C(\sigma)$是平行移动的

曲线$C(t)$沿切矢 $t^{a}=\left(\frac{\partial}{\partial t}\right)^{a}$, 沿$C(t)$导数定义为 $\frac{DV^{a}}{dt}=t^{b}D_{b}V^{a}$, 则平行移动 $\frac{DV^{a}}{dt}=0$

$\frac{DV^{a}}{dt}=t^{b}D_{b}V^{a}=t^{b}(\partial_{b}V^{a}+\Gamma^{a}_{bc}V^{c})$ 适配坐标系 $\frac{DV^{\mu}}{dt}=t^{\nu}\partial_{\nu}V^{\mu}+t^{\nu}\Gamma^{\mu}_{\nu\sigma}V^{\sigma}=\frac{dV^{\mu}}{dt}+\Gamma^{\mu}_{\nu\sigma}t^{\nu}t^{\sigma}$

测地线性质：

a. 设$\gamma(t)$为测地线, 则其重参数化$\gamma'(t')$ $(=\gamma(t))$的切矢T'^{a}满足$T'^{b}D_{b}T'^{a}=\alpha T'^{a}$. α为$\gamma(t)$上某函数.

proof:

$T^{a}=\left(\frac{\partial}{\partial t}\right)^{a}=\left(\frac{\partial}{\partial t'}\right)^{a}\frac{dt'}{dt}=\frac{dt'}{dt}T'^{a}$

从而 $0=T^{b}D_{b}T^{a}=\left(\frac{dt'}{dt}\right)T'^{b}D_{b}\left(\frac{dt'}{dt}T'^{a}\right)=\left(\frac{dt'}{dt}\right)^{2}T'^{b}D_{b}T'^{a}+\left(\frac{dt'}{dt}\right)T'^{b}T'^{a}D_{b}\left(\frac{dt'}{dt}\right)$

$=\left(\frac{dt'}{dt}\right)^{2}T'^{b}D_{b}T'^{a}+\left(\frac{dt'}{dt}\right)T'^{a}T'\left(\frac{dt'}{dt}\right)$

$=\left(\frac{dt'}{dt}\right)^{2}T'^{b}D_{b}T'^{a}+\left(\frac{dt'}{dt}\right)T'^{a}\frac{d}{dt'}\left(\frac{dt'}{dt}\right)$

$=\left(\frac{dt'}{dt}\right)^{2}T'^{b}D_{b}T'^{a}+T'^{a}\frac{d^{2}t'}{dt^{2}}$

$=0$

$$\Rightarrow T'^b \nabla_b T'^a = -T'^a \frac{d^2t'}{dt^2} \cdot \left(\frac{dt}{dt'}\right)^2 = \alpha T'^a, \quad \alpha = -\frac{d^2t'}{dt^2}\left(\frac{dt}{dt'}\right)^2$$

b. 设曲线 $\gamma(t)$ 切矢 T^a 满足 $T^b\nabla_b T^a = \alpha T^a$（$\alpha$ 为 $\gamma(t)$ 上函数），则存在 $t'=t'(t)$ 使 $\gamma(t') = \gamma(t)$ 为测地线.

c. 若 t 是某测地线的仿射参数，则该线的任一参数 t' 是仿射参数的充要条件是 $t'= at+b$，a, b 为常数且 $a\neq 0$.

d. 带联络的流形 (M, ∇_a) 的一点 p 及 p 的一个矢量 v^a 决定唯一的测地线，满足：

(1) $\gamma(0) = p$

(2) $\gamma(t)$ 在 p 的切矢等于 v^a.

e. 非类光测地线的线长参数必为仿射参数.

proof:

f. 设 g_{ab} 是流形 M 上的洛伦兹度规场，$p, q \in M$. 则 p, q 间的光滑类空（时）曲线为测地线当且仅当其线长取极值.

对任意时空中的类时联系的两点：

① 两点间最长线是类时测地线

② 两点间的类时测地线 未必是最长线（对闵氏空间一定是）

③ 两点间没有最短类时线.

12. 黎曼曲率张量

定义：$R^{\nu}{}_{\mu\kappa\lambda} = \partial_\kappa \Gamma^{\nu}_{\mu\lambda} - \partial_\lambda \Gamma^{\nu}_{\mu\kappa} + \Gamma^{\nu}_{\sigma\kappa}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\nu}_{\sigma\lambda}\Gamma^{\sigma}_{\mu\kappa}$

性质：a. 关于后两个指标反称，$R^{\nu}{}_{\mu\kappa\lambda} = -R^{\nu}{}_{\mu\lambda\kappa}$（定义）

b. 对无挠时空 $g_{\nu\rho} R^{\rho}{}_{\mu\kappa\lambda} = R_{\nu\mu\kappa\lambda} = -R_{\mu\nu\kappa\lambda}$，即关于前两个指标反称.

Proof:

$$R_{\rho\mu\kappa\lambda} = g_{\rho\nu} R^{\nu}{}_{\mu\kappa\lambda} = g_{\rho\nu}\left(\partial_\kappa \Gamma^{\nu}_{\mu\lambda} - \partial_\lambda \Gamma^{\nu}_{\mu\kappa} + \Gamma^{\nu}_{\sigma\kappa}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\nu}_{\sigma\lambda}\Gamma^{\sigma}_{\mu\kappa}\right)$$

$$\overset{\text{局部惯性系}}{=} g_{\rho\nu}\left(\partial_\kappa \Gamma^{\nu}_{\mu\lambda} - \partial_\lambda \Gamma^{\nu}_{\mu\kappa}\right)$$

$$= \frac{1}{2} g_{\rho\nu}\left\{\partial_\kappa\left[g^{\sigma\nu}(g_{\sigma\mu,\lambda} + g_{\sigma\lambda,\mu} - g_{\mu\lambda,\sigma})\right] - \partial_\lambda\left[g^{\sigma\nu}(g_{\sigma\mu,\kappa} + g_{\sigma\kappa,\mu} - g_{\mu\kappa,\sigma})\right]\right\}$$

$$= \frac{1}{2} g_{\rho\nu}\left[g^{\sigma\nu}{}_{,\kappa}(g_{\sigma\mu,\lambda} + g_{\sigma\lambda,\mu} - g_{\mu\lambda,\sigma}) + g^{\sigma\nu}(g_{\sigma\mu,\lambda\kappa} + g_{\sigma\lambda,\mu\kappa} - g_{\mu\lambda,\sigma\kappa})\right.$$
$$\left. - g^{\sigma\nu}{}_{,\lambda}(g_{\sigma\mu,\kappa} + g_{\sigma\kappa,\mu} - g_{\mu\kappa,\sigma}) - g^{\sigma\nu}(g_{\sigma\mu,\kappa\lambda} + g_{\sigma\kappa,\mu\lambda} - g_{\mu\kappa,\sigma\lambda})\right]$$

$$= \frac{1}{2}(g_{\rho\mu,\lambda\kappa} + g_{\rho\lambda,\mu\kappa} - g_{\mu\lambda,\rho\kappa}) - \frac{1}{2}(g_{\rho\mu,\kappa\lambda} + g_{\rho\kappa,\mu\lambda} - g_{\mu\kappa,\rho\lambda})$$

$$= \frac{1}{2}(g_{\rho\lambda,\mu\kappa} - g_{\mu\lambda,\rho\kappa} - g_{\rho\kappa,\mu\lambda} + g_{\mu\kappa,\rho\lambda})$$

从而有 $R_{\mu\rho\kappa\lambda} = \frac{1}{2}(g_{\mu\lambda,\rho\kappa} - g_{\rho\lambda,\mu\kappa} - g_{\mu\kappa,\rho\lambda} + g_{\rho\kappa,\mu\lambda}) = -R_{\rho\mu\kappa\lambda}$.

c、$R_{\mu\nu\kappa\lambda}$ 前一对指标与后一对指标对称，$R_{\mu\nu\kappa\lambda} = R_{\kappa\lambda\mu\nu}$.

Proof:

局部惯性系中，$R_{\mu\nu\kappa\lambda} = \frac{1}{2}(g_{\mu\lambda,\nu\kappa} + g_{\nu\kappa,\mu\lambda} - g_{\mu\kappa,\nu\lambda} - g_{\nu\lambda,\mu\kappa})$

$\Rightarrow \quad R_{\kappa\lambda\mu\nu} = \frac{1}{2}(g_{\kappa\nu,\lambda\mu} + g_{\lambda\mu,\kappa\nu} - g_{\kappa\mu,\lambda\nu} - g_{\lambda\nu,\kappa\mu}) = R_{\mu\nu\kappa\lambda}$

里奇恒等式 $R^{\nu}_{\ (\mu\kappa\lambda)} = 0$ 或 $R^{\nu}_{\ \mu\kappa\lambda} + R^{\nu}_{\ \kappa\lambda\mu} + R^{\nu}_{\ \lambda\mu\kappa} = 0$

Proof：在联络系数各分量均为0的局部惯性系有

$R^{\nu}_{\ \mu\kappa\lambda} = \partial_\kappa \Gamma^{\nu}_{\mu\lambda} - \partial_\lambda \Gamma^{\nu}_{\mu\kappa} = 2\Gamma^{\nu}_{\mu[\lambda,\kappa]}$

从而 $R^{\nu}_{\ (\mu\kappa\lambda)} = 2\Gamma^{\nu}_{(\mu[\lambda,\kappa])} = 0$.

里奇曲率张量：$R_{\mu\lambda} = R^{\nu}_{\ \mu\nu\lambda}$（一、三缩并）$= -R^{\nu}_{\ \mu\lambda\nu}$

$R^{\nu}_{\ \nu\kappa\lambda} = g^{\nu\mu} R_{\nu\mu\kappa\lambda} = 0$，即一、二指标缩并为0.

关于 μ,λ 对称：$R_{\mu\lambda} = R_{\lambda\mu}$.

里奇标量：$R = g^{\mu\lambda} R_{\mu\lambda}$

爱因斯坦张量（关于 μ,ν 对称张量，有 $\frac{n(n+1)}{2}$ 个独立分量）

$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

比安基恒等式：$R^{\mu}_{\ \nu(\kappa\lambda;\sigma)} = 0 \Leftrightarrow R^{\mu}_{\ \nu\kappa\lambda;\sigma} + R^{\mu}_{\ \nu\lambda\sigma;\kappa} + R^{\mu}_{\ \nu\sigma\kappa;\lambda} = 0$

Proof：在局部惯性系中，

$R^{\mu}_{\ \nu\kappa\lambda,\sigma} = \Gamma^{\mu}_{\nu\lambda,\kappa\sigma} - \Gamma^{\mu}_{\nu\kappa,\lambda\sigma} = 2\Gamma^{\mu}_{\nu[\lambda,\kappa]\sigma}$

$\Rightarrow R^{\mu}_{\ \nu(\kappa\lambda;\sigma)} = R^{\mu}_{\ \nu(\kappa\lambda,\sigma)} = 2\Gamma^{\mu}_{\nu([\lambda,\kappa]\sigma)} = 0$

$G^{\mu}_{\ \lambda;\mu} = 0$

Proof：由比安基恒等式有 $R^{\mu}_{\ \nu\kappa\lambda;\sigma} + R^{\mu}_{\ \nu\sigma\kappa;\lambda} + R^{\mu}_{\ \nu\lambda\sigma;\kappa} = 0$

μ 与 κ 收缩有 $\quad R_{\nu\lambda;\sigma} + \underbrace{R^{\mu}_{\ \nu\sigma\mu;\lambda}}_{} + R^{\mu}_{\ \nu\lambda\sigma;\mu} = 0$

$-R^{\mu}_{\ \nu\mu\sigma;\lambda}$

$= -R_{\nu\sigma;\lambda}$

$\Rightarrow R_{\nu\lambda;\sigma} - R_{\nu\sigma;\lambda} + R^{\mu}_{\ \nu\lambda\sigma;\mu} = 0$

与 $g^{\nu\sigma}$ 收缩有 $\ 0 = R^{\sigma}_{\ \lambda;\sigma} - R_{;\lambda} + R^{\mu}_{\ \lambda;\mu} = 2G^{\mu}_{\ \lambda;\mu}$.

13、killing 矢量场

killing 方程：$\mathcal{L}_\xi g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$ 或 $\nabla_{(a}\xi_{b)} = 0$ 或 $\nabla_a \xi_b = \nabla_{[a}\xi_{b]}$

Proof:

$0 = \mathcal{L}_\xi g_{\mu\nu} = \underbrace{\xi^\lambda \nabla_\lambda g_{\mu\nu}}_{0} + g_{\lambda\nu}\nabla_\mu \xi^\lambda + g_{\mu\lambda}\nabla_\nu \xi^\lambda = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$

性质：设 ξ^a 为 killing 矢量场，T^a 为测地线切矢，有 $T^a\nabla_a(T^b\xi_b) = 0$，即 $T^b\xi_b$ 在测地线上为常数.

Proof: $T^a\nabla_a(T^b\xi_b) = (T^a\nabla_a T^b)\xi_b + T^aT^b\nabla_a\xi_b = T^{(a}T^{b)}\nabla_{[a}\xi_{b]} = 0$

定理：若存在坐标系 $\{x^\mu\}$ 使 g_{ab} 全部分量满足 $\frac{\partial g_{\mu\nu}}{\partial x^1}=0$，则 $(\frac{\partial}{\partial x^1})^a$ 为坐标域的 Killing 方程

例：2维平面 $ds^2 = dx^2 + dy^2$

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} \frac{\partial g_{\mu\nu}}{\partial x} = 0 \\ \frac{\partial g_{\mu\nu}}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} \xi_1^a = (\frac{\partial}{\partial x})^a \\ \xi_2^a = (\frac{\partial}{\partial y})^a \end{cases}$$

Killing 方程有

$$\begin{cases} \xi_{x;y} + \xi_{y;x} = 0 \\ \xi_{x;x} = \xi_{y;y} = 0 \end{cases} \Rightarrow \begin{cases} \xi_{x,y} + \xi_{y,x} = 0 \\ \xi_{x,x} = \xi_{y,y} = 0 \end{cases} \text{有} \begin{cases} \xi_x = -by + a_x \\ \xi_y = bx + a_y \end{cases} \Rightarrow \begin{cases} \xi_1^\mu = (1,0) \\ \xi_2^\mu = (0,1) \\ \xi_3^\mu = (-y,x) \end{cases}$$

故
$$\begin{cases} \xi_1^a = (\frac{\partial}{\partial x})^a \\ \xi_2^a = (\frac{\partial}{\partial y})^a \\ \xi_3^a = -y(\frac{\partial}{\partial x})^a + x(\frac{\partial}{\partial y})^a \end{cases}$$

2维单位球面，$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \Rightarrow \frac{\partial g_{\mu\nu}}{\partial\varphi} = 0 \Rightarrow \xi_1^a = (\frac{\partial}{\partial\varphi})^a$$

Killing 方程 $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$

$\xi^\theta{}_{;\varphi} = \nabla_\varphi \xi^\theta = \xi^\theta{}_{,\varphi} + \Gamma^\theta_{\varphi\varphi}\xi^\varphi = \xi^\theta{}_{,\varphi} - \sin\theta\cos\theta\,\xi^\varphi$

$\xi^\varphi{}_{;\theta} = \nabla_\theta\xi^\varphi = \xi^\varphi{}_{,\theta} + \Gamma^\varphi_{\varphi\theta}\xi^\varphi = \xi^\varphi{}_{,\theta} + \cot\theta\,\xi^\varphi$

$\mu=\nu=\theta$; $\xi_{\theta;\theta} = 0 \Rightarrow \xi^\theta{}_{,\theta} = 0$

$\mu=\theta, \nu=\varphi$: $\xi_{\theta;\varphi} + \xi_{\varphi;\theta} = 0 \Rightarrow g_{\theta\theta}\xi^\theta{}_{;\varphi} + g_{\varphi\varphi}\xi^\varphi{}_{;\theta} = \xi^\theta{}_{,\varphi} + \Gamma^\theta_{\varphi\varphi}\xi^\varphi + \sin^2\theta\,\xi^\varphi{}_{,\theta} + \sin^2\theta\,\Gamma^\varphi_{\varphi\theta}\xi^\varphi = 0$

$\Rightarrow \xi^\theta{}_{,\varphi} + \xi^\varphi{}_{,\theta}\sin^2\theta = 0$

$\xi_{\varphi;\varphi} = 0 \Rightarrow g_{\varphi\varphi}\xi^\varphi{}_{;\varphi} = 0 \Rightarrow \sin^2\theta\,\xi^\varphi{}_{,\varphi} + \sin^2\theta\,\Gamma^\varphi_{\theta\varphi}\xi^\theta = 0 \Rightarrow \xi^\varphi{}_{,\varphi}\sin\theta + \xi^\theta\cos\theta = 0$

$\xi^\varphi{}_{;\varphi} = \xi^\varphi{}_{,\varphi} + \Gamma^\varphi_{\theta\varphi}\xi^\theta = \xi^\varphi{}_{,\varphi} + \cot\theta\,\xi^\theta$

$$\Rightarrow \begin{cases} \xi^\theta = A\sin(\varphi-\alpha) \\ \xi^\varphi = A\cos(\varphi-\alpha)\cot\theta + b \end{cases}$$

取三个线性独立的 Killing 场，取 $(A,\alpha,b) = (-1,0,0), (-1,\frac{\pi}{2},0), (0,0,1)$

$\xi_1^\mu = (-\sin\varphi, -\cos\varphi\cot\theta)$

$\xi_2^\mu = (\cos\varphi, -\sin\varphi\cot\theta)$ $\qquad \frac{\partial}{\partial x^1} = \frac{\partial}{\partial\theta}, \frac{\partial}{\partial x^2} = \frac{\partial}{\partial\varphi}$

$\xi_3^\mu = (0, 1)$

$\xi_1 = \xi_1^\mu\partial_\mu = -\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}$

$\xi_2 = \xi_2^\mu\partial_\mu = \cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}$

$\xi_3 = \xi_3^\mu\partial_\mu = \frac{\partial}{\partial\varphi}$

2维闵氏时空 $ds^2 = -dt^2 + dx^2$

2维闵氏时空 $g_{tt}=-1,\ g_{xx}=1 \Rightarrow g^{tt}=-1,\ g^{xx}=1$

$$(g_{\mu\nu})=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\quad \begin{cases}\dfrac{\partial g_{\mu\nu}}{\partial t}=0\\[2mm] \dfrac{\partial g_{\mu\nu}}{\partial x}=0\end{cases} \Rightarrow \begin{matrix}\xi_1^a=\left(\dfrac{\partial}{\partial t}\right)^a\\[2mm] \xi_2^a=\left(\dfrac{\partial}{\partial x}\right)^a\end{matrix}$$

Killing方程 $\xi_{\mu;\nu}+\xi_{\nu;\mu}=0$

$\mu=x,\ \nu=t$ 有 $\xi_{x;t}+\xi_{t;x}=0$ $\qquad \xi_{x,t}+\xi_{t,x}=0 \rightarrow g_{xx}\xi^x{}_{,t}+g_{tt}\xi^t{}_{,x}=\xi^x{}_{,t}-\xi^t{}_{,x}=0$

$\mu=\nu=x$ 有 $\xi_{x;x}=0$ $\xrightarrow[\text{变为普通导数}]{\text{闵氏空间下克氏符}}$ $\xi_{x,x}=0 \rightarrow g_{xx}\xi^x{}_{,x}=\xi^x{}_{,x}=0$

$\mu=\nu=t$ 有 $\xi_{t;t}=0$ $\qquad \xi_{t,t}=0 \rightarrow g_{tt}\xi^t{}_{,t}=\xi^t{}_{,t}=0$

通解为

$$\begin{cases}\xi^t=bx+a^t\\ \xi^x=bt+a^x\end{cases} \Rightarrow \begin{cases}\xi_1^\mu=(1,0)\\ \xi_2^\mu=(0,1)\\ \xi_3^\mu=(x,t)\end{cases} \Rightarrow \xi_3^a=(\xi_3^\mu\partial_\mu)^a=x\left(\frac{\partial}{\partial t}\right)^a+t\left(\frac{\partial}{\partial x}\right)^a$$

二、广义相对论的经典检验

①水星近日点的进动

②光线偏折 $\qquad$ 验证了 $\left(\frac{GM}{c^2R}\right)^2$ 的二级效应：①

③引力红移 $\qquad$ 验证了场方程：①②③

④雷达回波延迟 $\qquad$ 验证了 $\frac{GM}{c^2R}$ 的一级效应：②③

引力红移

定义：由于光源与观察者静止于静态引力场中不同位置而引起的光谱红移

静止于 r_0 处的观察者接收到发自静止于 r_s 处的光源发出的光的红移为

波长：$z=\sqrt{\dfrac{1-\frac{2GM}{r_0}}{1-\frac{2GM}{r_0}}}-1$

频率：$\dfrac{\text{接收频率}}{\text{发射频率}}=\dfrac{\omega_0}{\omega_s}=\sqrt{\dfrac{1-\frac{2GM}{r_s}}{1-\frac{2GM}{r_0}}}=\sqrt{\dfrac{-g_{tt}(r_s)}{-g_{tt}(r_0)}}$

爱因斯坦场方程：$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\frac{8\pi G}{c^4}T_{\mu\nu}$

三、引力波

线性引力场方程

设 $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$，$\|h_{\mu\nu}\|<1$，则有 $\Gamma^\lambda_{\mu\nu}\approx\frac{1}{2}\eta^{\lambda\sigma}(h_{\mu\sigma,\nu}+h_{\sigma\nu,\mu}-h_{\mu\nu,\sigma})$

从而 $\Gamma^\lambda=g^{\mu\nu}\Gamma^\lambda_{\mu\nu}=(\eta^{\mu\nu}+h^{\mu\nu})\frac{1}{2}\eta^{\lambda\sigma}(h_{\mu\sigma,\nu}+h_{\sigma\nu,\mu}-h_{\mu\nu,\sigma})$

$$\approx\frac{1}{2}\eta^{\mu\nu}(h^\lambda{}_{\mu,\nu}+h^\lambda{}_{\nu,\mu}-\eta^{\lambda\sigma}h_{\mu\nu,\sigma})$$

$$=h^{\lambda\nu}{}_{,\nu}-\frac{1}{2}\eta^{\lambda\nu}h_{,\nu}=:\bar{h}^{\lambda\nu}{}_{,\nu}$$

$$
\begin{aligned}
R_{\mu\nu} &\approx \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} \\
&= \tfrac{1}{2}\eta^{\lambda\sigma}\left(h_{\mu\sigma,\nu\lambda} + h_{\sigma\nu,\mu\lambda} - h_{\mu\nu,\sigma\lambda} - h_{\mu\sigma,\lambda\nu} - h_{\sigma\lambda,\mu\nu} + h_{\mu\lambda,\sigma\nu}\right) \\
&= \tfrac{1}{2}\eta^{\lambda\sigma}\left(h_{\sigma\nu,\mu\lambda} - h_{\mu\nu,\sigma\lambda} - h_{\sigma\lambda,\mu\nu} + h_{\mu\lambda,\sigma\nu}\right) \\
&= \tfrac{1}{2}\left(\eta^{\lambda\sigma}h_{\sigma\nu,\mu\lambda} - \eta^{\lambda\sigma}h_{\mu\nu,\sigma\lambda} - \eta^{\lambda\sigma}h_{\sigma\lambda,\mu\nu} + \eta^{\lambda\sigma}h_{\mu\lambda,\sigma\nu}\right) \\
&= \tfrac{1}{2}\left(\eta^{\lambda\sigma}h_{\sigma\nu,\mu\lambda} - \Box h_{\mu\nu} - h_{,\mu\nu} + \eta^{\lambda\sigma}h_{\mu\lambda,\sigma\nu}\right) \\
&= -\tfrac{1}{2}\left(\Box h_{\mu\nu} - \eta_{\nu\rho}h^{\rho\lambda}{}_{,\mu\lambda} + \Box h_{\mu\nu} - \eta_{\mu\rho}h^{\rho\sigma}{}_{,\sigma\nu}\right) \\
&= -\tfrac{1}{2}\left(\Box h_{\mu\nu} - \eta_{\nu\rho}\bar{h}^{\rho\lambda}{}_{,\mu\lambda} - \eta_{\mu\rho}\bar{h}^{\rho\lambda}{}_{,\nu\lambda}\right)
\end{aligned}
$$

$$R = \eta^{\mu\nu}R_{\mu\nu} = -(\Box h - h^{\rho\lambda}{}_{,\lambda\rho})$$

$\Rightarrow$

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = -\tfrac{1}{2}\left(\Box \bar{h}_{\mu\nu} - \eta_{\nu\rho}\bar{h}^{\rho\lambda}{}_{,\lambda\mu} - \eta_{\mu\rho}\bar{h}^{\rho\lambda}{}_{,\lambda\nu} + \eta_{\mu\nu}\bar{h}^{\lambda\sigma}{}_{,\lambda\sigma}\right) = 8\pi G T_{\mu\nu}$$

当 $\bar{h}^{\lambda\nu}{}_{,\nu} = 0$ 有 $\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$

引力波

无迹横波，有两个独立分量 $h_{11}=-h_{22}$，$h_{12}=h_{21}$

引力波量子化后可视为静止质量为0，自旋为2的引力子。

引力辐射是四极辐射，不存在单极和偶极辐射。四极辐射强度与频率的6次方、转动惯量的平方成正比。

四、星体内部结构与黑洞物理

辐射压<引力
- 白矮星：电子简并压 平衡引力，质量上限：钱德拉塞卡极限 1.24-1.44 $M_\odot$
- 中子星：中子简并压：纯中子星质量上限：$M_{max}=0.7M_\odot$
- 夸克星：夸克简并压
- 黑洞：

稳态时空：存在类时Killing矢量场，即存在时间坐标t，使 $g_{\mu\nu}$ 不依赖于t.

静态时空：存在超曲面正交的Killing矢量场，或称没有 $dt\,dx^i$ 项的稳态时空。

史瓦西解：

$$ds^2=-\left(1-\frac{2GM}{r}\right)dt^2+\left(1-\frac{2GM}{r}\right)^{-1}dr^2+\underbrace{r^2d\theta^2+r^2\sin^2\theta\, d\varphi^2}_{r^2d\Omega^2}=0$$

- $r=0$：内禀奇异性不可消除
- $r=2M$：坐标奇异性，可消除

• 无限红移面：$r=2M$ 时 $\omega\to 0$，$\lambda\to\infty$

• 静界：$r=2M$，$r>2M$ 时观察者可静止于时空中，即时空曲线 $r,\theta,\varphi=$ 常数为一条类时世界线。

• 单向膜与视界：

$r>2M$：$r=C$ 是一个类时超曲面，信号可双向通行

$r=2M$：单向膜 类光超曲面，

$r<2M$：两界↓视界 类空超曲面。

信号只进不出

• 时空坐标互换

	$1-\frac{2M}{r}$	时间坐标	空间坐标
$r>2M$	+	t	r
$r<2M$	−	r	t

彭罗斯图

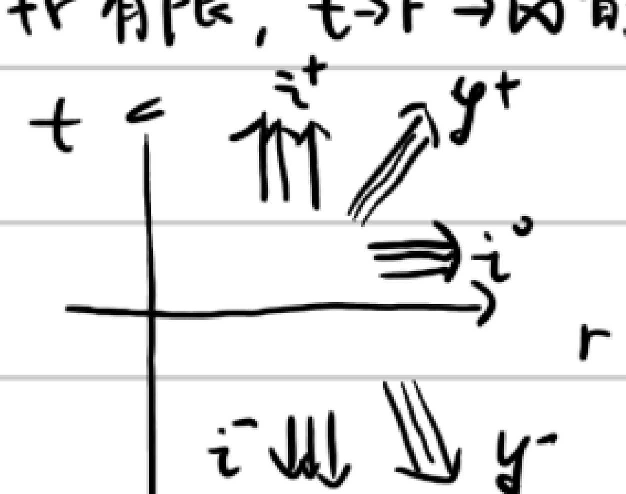
r有限，$t\to+\infty$ 的极限点：类时未来无穷远，i^+

r有限，$t\to-\infty$ 的极限点：类时过去无穷远，i^-

t有限，$r\to+\infty$ 的极限点：类空无穷远，i^0

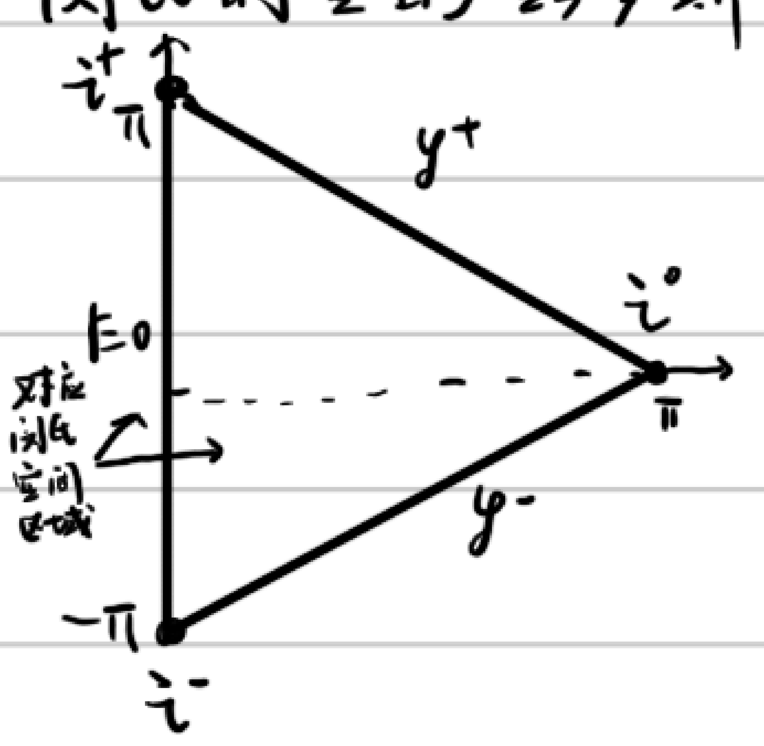
$t-r$ 有限，$t+r\to+\infty$ 的极限点，类光未来无穷远 $\mathscr{I}^+$

$t+r$ 有限，$t-r\to-\infty$ 的极限点，类光过去无穷远 $\mathscr{I}^-$



闭合时空的无穷远点.

闵氏时空的彭罗斯图



闵氏时空彭罗斯图上每个点代表一个球面
类空无穷远点都变成一点。

Kerr-Newman解：描述一个转动带电星体的外部引力场

$M\neq0$, $J\neq0$, $Q=0$ 回到 Kerr时空

$M\neq0$, $J=0$, $Q\neq0$ 回到 R-N时空（球对称电真空解）

$M\neq0$, $J=0$, $Q=0$ 回到史瓦西时空。

若干定理：

奇点定理：若因果律和爱因斯坦场方程成立，且物质场满足适当的能量条件，则大质量恒星坍缩必导致时空奇异性的出现。

宇宙监督假说：一物体的完全引力坍缩总是形成黑洞而不是形成裸奇异性。

无毛定理：渐近平直时空中，引力与物质场最小耦合时，稳态黑洞仅由质量M、电荷Q与自转角动量$J=aM$三个参量确定。

面积不减定理：假设 ①宇宙监督假说成立 ②弱能量条件 $T_{00}\geq0$ 则所有黑洞视界的总面积在未来永不减少。

黑洞热力学定律

第零定律：稳态黑洞视界表面温度为常数。

第一定律：$\delta M = T\delta S + \Omega_H \delta J + \phi_H \delta q$

第二定律：$\delta S \geq 0$

第三定律：不能通过有限物理过程把黑洞视界表面温度变为0。

霍金辐射：黑洞不是完全黑的，存在向外的辐射，具有标准的黑体谱。黑体谱的温度正比于黑洞的表面引力，即 $T_H = \frac{\kappa}{2\pi k_B}$

五. 宇宙学初步

宇宙学原理：宇观尺度上任何时刻，宇宙的3维空间都是均匀各向同性的。

FLRW度规：$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$，$a$为尺度因子，$k=\pm1, 0$.

Hubble定律：河外星云退行速度 $v = Hd$，d是河外星云到地球的距离，H为Hubble常数。

$z = d_L H_0$，z为宇宙学红移，d_L为光度距离，H_0为Hubble常数。

Friedman方程

$\dot{a}^2+K=\frac{8\pi G}{3}\rho a^2 \Leftrightarrow H^2+\frac{K}{a^2}=\frac{8\pi G}{3}\rho$, $H=\frac{\dot{a}}{a}$ 为 Hubble参数. ρ, p 分别为固有能量密度与各向同性压强.

$\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}(\rho+3p)$

能动量守恒: $\frac{\dot{\rho}}{\rho+p}=-3\frac{\dot{a}}{a}$

物态方程: $p=w\rho$ (理想流体)

$\Rightarrow \frac{\dot{\rho}}{\rho+w\rho}=-3\frac{\dot{a}}{a}$

$\frac{\dot{\rho}}{\rho(w+1)}=-3\frac{\dot{a}}{a} \Rightarrow d\ln\rho=-3(w+1)\,d\ln a$

$\ln\rho=-3(w+1)\ln a$, 即 $\rho=\rho_0 a^{-3(w+1)}$

$w=0$, 物质(重子,暗物质) $\rho\propto a^{-3}$

$w=\frac{1}{3}$ 辐射(光子,无质量中微子,相对论性粒子) $\rho\propto a^{-4}$

$w=-1$ 暗能量(宇宙学常数) $\rho\propto\rho_0$

考虑 $K=0$, $H^2=\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3}\rho$

$$a\propto\begin{cases} t^{\frac{2}{3(1+w)}}, & w\neq-1 \\ e^{Ht}, & w=-1 \end{cases}$$

$t^{\frac{2}{3}}$ 物质 ($w=0$)

$t^{\frac{1}{2}}$ 辐射 ($w=\frac{1}{3}$)

暗能量

由于 $\dot{a}>0$ 而 $\ddot{a}$ 满足

$$\ddot{a}\begin{cases} >0, & w<-\frac{1}{3} \rightarrow \text{加速膨胀} \\ <0, & w>-\frac{1}{3} \rightarrow \text{减速膨胀} \end{cases}$$

能量条件:

- 弱能量条件: 任一瞬时观者(p, Z^a)测得的能量密度非负. $T_{ab}u^a u^b\geq 0$, $\forall$类时矢量场 u^a.
- 强能量条件: $T_{ab}Z^aZ^b\geq-\frac{1}{2}T$, 单位类时矢量场 Z^a.
- 主能量条件: 任一瞬时观者(p, Z^a)测得的4动量密度 $W^a=-T^a{}_b Z^b$ 是指向未来的类时或类光矢量.

物理解释为: 物质场能量流动速率小于或等于光速.

最小耦合原理: 从平直时空过渡到闵氏时空只要把闵氏度规变为弯曲时空的度规场, 把偏导数变为协变导数, 即

$\eta_{\mu\nu}\rightarrow g_{\mu\nu}$, $\partial_\mu\rightarrow\nabla_\mu$.

稳定圆轨道:

光的圆轨道是不稳定的。

对质点: $\frac{d^2V_{eff}}{dr^2}=2GMr\frac{r-6GM}{r-3GM}>0 \Rightarrow r>6GM$ 或 $r<3GM$(不存在圆轨道,舍)