广义相对论 (2024 秋)

习题解答

2024 年 12 月 31 日

Exercise 1 :

证明：设 $A^\mu(x)$ 和 $B^\mu(x)$ 是两个光滑逆变矢量场, 证明其对易子:

$$[A,B]^\mu \equiv A^\nu\partial_\nu B^\mu - B^\nu\partial_\nu A^\mu$$

也是一个逆变矢量场.

证明. 由于

$$\begin{aligned}
[A',B']^\mu &= A'^\nu\partial'_\nu B'^\mu - B'^\nu\partial'_\nu A'^\mu \\
&= \frac{\partial x'^\nu}{\partial x^\rho}A^\rho\frac{\partial x^\sigma}{\partial x'^\nu}\partial_\sigma\left(\frac{\partial x'^\mu}{\partial x^\tau}B^\tau\right) - \frac{\partial x'^\nu}{\partial x^\rho}B^\rho\frac{\partial x^\sigma}{\partial x'^\nu}\partial_\sigma\left(\frac{\partial x'^\mu}{\partial x^\tau}A^\tau\right) \\
&= A^\rho\delta^\sigma{}_\rho\partial_\sigma\left(\frac{\partial x'^\mu}{\partial x^\tau}B^\tau\right) - B^\rho\delta^\sigma{}_\rho\partial_\sigma\left(\frac{\partial x'^\mu}{\partial x^\tau}A^\tau\right) \\
&= A^\sigma\partial_\sigma\left(\frac{\partial x'^\mu}{\partial x^\tau}B^\tau\right) - B^\sigma\partial_\sigma\left(\frac{\partial x'^\mu}{\partial x^\tau}A^\tau\right) \\
&= \frac{\partial x'^\mu}{\partial x^\tau}\left(A^\sigma\partial_\sigma B^\tau - B^\sigma\partial_\sigma A^\tau\right) + \underbrace{A^\sigma(\partial_\sigma\partial_\tau x'^\mu)B^\tau - B^\sigma(\partial_\tau\partial_\sigma x'^\mu)A^\tau}_{=2\partial_{(\sigma}\partial_{\tau)}x'^\mu A^{[\sigma}B^{\tau]}=0} \\
&= \frac{\partial x'^\mu}{\partial x^\tau}\left(A^\sigma\partial_\sigma B^\tau - B^\sigma\partial_\sigma A^\tau\right) = \frac{\partial x'^\mu}{\partial x^\tau}[A,B]^\tau,
\end{aligned} \tag{1}$$

所以 $[A,B]^\mu \equiv A^\nu\partial_\nu B^\mu - B^\nu\partial_\nu A^\mu$ 也是一个逆变矢量场. □

Exercise 2 :

(a) 令 $S^{\mu\nu} = V^\mu V^\nu$, 对 $\forall$ 矢量场 V^μ, 证明：$S^{(\mu\nu)} = S^{\mu\nu}$；

(b) 证明 $T_{\mu\nu} = T_{[\mu\nu]}$ 的充要条件是：$T_{\mu\nu}V^\mu V^\nu = 0$ 对 $\forall$ 矢量 V^μ 都成立.

证明.

对 **(a)**

$$S^{(\mu\nu)} = \frac{1}{2}(S^{\mu\nu}+S^{\nu\mu}) = \frac{1}{2}(V^\mu V^\nu + V^\nu V^\mu) = \frac{1}{2}(V^\mu V^\nu + V^\mu V^\nu) = V^\mu V^\nu = S^{\mu\nu}. \tag{2}$$

对 **(b)**, **充分性:**

因为 $\forall V^\mu, V^\nu$ 有

$$0 = T_{\mu\nu}V^\mu V^\nu = T_{\nu\mu}V^\nu V^\mu = T_{\nu\mu}V^\mu V^\nu \Rightarrow (T_{\mu\nu}+T_{\nu\mu})V^\mu V^\nu = 0, \tag{3}$$

所以 $T_{\mu\nu}+T_{\nu\mu}=0$, 从而有

$$T_{[\mu\nu]}=\frac{1}{2}(T_{\mu\nu}-T_{\nu\mu})=\frac{1}{2}\cdot 2T_{\mu\nu}=T_{\mu\nu}. \tag{4}$$

必要性:

$$T_{\mu\nu}V^{\mu}V^{\nu}=T_{[\mu\nu]}V^{(\mu}V^{\nu)}=0. \tag{5}$$

□

Exercise 3 :

设 $C(t)$ 是曲线,$x^{\mu}(t)$ 是其在某坐标下的参数式,$p=C(t_1)$, $q=C(t_2)$, 则从 p 到 q 的线长为:

$$l=\int_{t_1}^{t_2}\sqrt{g_{\mu\nu}\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt}}\,dt$$

请给出极值曲线所满足的方程.（注：此处取类空曲线）

证明. 设 $C'(t)$ 是与 $C(t)$ 无限靠近、起点与终点重合的一条类空曲线, 其参数式为 $x'^{\mu}(t)$, 满足

$$\begin{cases} x'^{\mu}(t_1)=x^{\mu}(t_1) \\ x'^{\mu}(t_2)=x^{\mu}(t_2) \end{cases} \tag{6}$$

而且 $x^{\mu}(t)$ 的变分 $\delta x^{\mu}(t)=x'^{\mu}(t)-x^{\mu}(t)$ 要多小有多小. 于是就有

$$\delta g_{\mu\nu}=g_{\mu\nu}[x^{\sigma}(t)+\delta x^{\sigma}(t)]-g_{\mu\nu}[x^{\sigma}(t)]=\frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\delta x^{\sigma}(t). \tag{7}$$

由于求导和变分符号可交换, 所以有

$$\delta\left(\dot{x}^{\mu}\right)=\frac{\mathrm{d}(\delta x^{\mu})}{\mathrm{d}t}. \tag{8}$$

对线长 l 左右两边取变分, 有

$$\delta l=\frac{1}{2}\int_{t_1}^{t_2}\left(g_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}\right)^{-1/2}\delta\left[g_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}\right]\mathrm{d}t, \tag{9}$$

考虑到

$$\begin{aligned}\delta\left[g_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}\right]&=\delta g_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}+g_{\mu\nu}\delta\left(\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\right)\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}+g_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\delta\left(\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}\right)\\&\overset{(8),(9)}{=\!=\!=}\frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\delta x^{\sigma}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}+g_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta x^{\mu}\right)\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}+g_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}(\delta x^{\nu})\\&=\frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\delta x^{\sigma}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}+g_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta x^{\mu}\right)\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}+g_{\nu\mu}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}(\delta x^{\mu})\\&\overset{g_{\mu\nu}=g_{\nu\mu}}{=\!=\!=\!=}\frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\delta x^{\sigma}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}+2g_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta x^{\mu}\right)\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t},\end{aligned} \tag{10}$$

于是就有

$$
\begin{aligned}
\delta l &= \frac{1}{2}\int_{t_1}^{t_2}\left(g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right)^{-1/2}\delta\left[g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right]\mathrm{d}t\\
&= \frac{1}{2}\int_{t_1}^{t_2}\left(g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right)^{-1/2}\left[\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\delta x^\sigma\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}+2g_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta x^\mu\right)\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right]\mathrm{d}t\\
&= \int_{t_1}^{t_2}\left(g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right)^{-1/2}\left[\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\delta x^\sigma\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}+g_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta x^\mu\right)\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right]\mathrm{d}t\\
&= \int_{t_1}^{t_2}\left(g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right)^{-1/2}\left[\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\delta x^\sigma\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}+\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\delta x^\mu\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right)\delta x^\mu\right]\mathrm{d}t.
\end{aligned}
\tag{11}
$$

对于非类光曲线, 总可以选取参数使得线长归一, 即 $g_{\mu\nu}\dfrac{\mathrm{d}x^\mu}{\mathrm{d}t}\dfrac{\mathrm{d}x^\nu}{\mathrm{d}t}=1$, 于是就有

$$
\begin{aligned}
\delta l &= \int_{t_1}^{t_2}\left[\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\delta x^\sigma\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}-\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right)\delta x^\mu\right]\mathrm{d}t+g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\delta x^\mu\bigg|_{t_1}^{t_2}\\
&\overset{\delta x^\mu|_{C(t_1)}=0}{\underset{\delta x^\mu|_{C(t_2)}=0}{=\!=\!=\!=\!=}} \int_{t_1}^{t_2}\left[\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\delta x^\sigma\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}-\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right)\delta x^\mu\right]\mathrm{d}t\\
&= \int_{t_1}^{t_2}\left[\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}-\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right)\right]\delta x^\sigma\mathrm{d}t
\end{aligned}
\tag{12}
$$

上式为零必然导致

$$
\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}-\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{\sigma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right)=\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}-g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}t^2}-\frac{\partial g_{\sigma\nu}}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}=0,\tag{13}
$$

式 13两边同乘 $g^{\rho\sigma}$ 得到

$$
\begin{aligned}
&-g^{\rho\sigma}g_{\sigma\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}t^2}+\frac{1}{2}g^{\rho\sigma}\left[\frac{\partial g_{\mu\nu}}{\partial x^\sigma}-2\frac{\partial g_{\sigma\nu}}{\partial x^\mu}\right]\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\\
&=-\frac{\mathrm{d}^2x^\rho}{\mathrm{d}t^2}+\frac{1}{2}g^{\rho\sigma}\left(g_{\mu\nu,\sigma}-g_{\sigma\nu,\mu}-g_{\mu\sigma,\nu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\\
&=-\frac{\mathrm{d}^2x^\rho}{\mathrm{d}t^2}-\Gamma^\rho{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}=0,
\end{aligned}
\tag{14}
$$

即

$$
\frac{\mathrm{d}^2x^\rho}{\mathrm{d}t^2}+\Gamma^\rho{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}=0,\tag{15}
$$

这说明极值曲线满足测地线方程. □

Exercise 4 :

二维欧氏空间的度规为：$ds^2 = dx^2 + dy^2$, 设 $f(x, y)$ 为其上的标量场, 请计算：$d\,{}^*df$. (其中 d 代表外微分,* 为 Hodge start)

解: 对于标量场 f, 有

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y, \tag{16}$$

利用 ${}^*\mathrm{d}x = \mathrm{d}y$ 和 ${}^*\mathrm{d}y = -\mathrm{d}x$ 可以得到

$${}^*\mathrm{d}f = \frac{\partial f}{\partial x}({}^*\mathrm{d}x) + \frac{\partial f}{\partial y}({}^*\mathrm{d}y) = \frac{\partial f}{\partial x}\mathrm{d}y - \frac{\partial f}{\partial y}\mathrm{d}x, \tag{17}$$

从而有

$$\begin{aligned} \mathrm{d}^*\mathrm{d}f &= \mathrm{d}\left(\frac{\partial f}{\partial x}\mathrm{d}y - \frac{\partial f}{\partial y}\mathrm{d}x\right) \\ &= \mathrm{d}\left(\frac{\partial f}{\partial x}\right)\wedge \mathrm{d}y - \mathrm{d}\left(\frac{\partial f}{\partial y}\right)\wedge \mathrm{d}x \\ &= \left(\frac{\partial^2 f}{\partial x^2}\mathrm{d}x + \frac{\partial^2 f}{\partial x \partial y}\mathrm{d}y\right)\wedge \mathrm{d}y - \left(\frac{\partial^2 f}{\partial y^2}\mathrm{d}y + \frac{\partial^2 f}{\partial y \partial x}\mathrm{d}x\right)\wedge \mathrm{d}x \\ &= \frac{\partial^2 f}{\partial x^2}\mathrm{d}x\wedge \mathrm{d}y - \frac{\partial^2 f}{\partial y^2}\mathrm{d}y\wedge \mathrm{d}x \\ &= \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}x\wedge \mathrm{d}y, \end{aligned} \tag{18}$$

上面用到了 $\mathrm{d}^2 = 0$ 以及 $\mathrm{d}y\wedge \mathrm{d}x = -\mathrm{d}x\wedge \mathrm{d}y$.

Exercise 5 :

对于 1-形式场 ω_a, 请通过计算验证：($\mathcal{L}$ 为李导数)

$$\mathcal{L}_v \circ d = d \circ \mathcal{L}_v$$

解: 由于

$$\mathrm{d}\boldsymbol{\omega}(v) = \mathrm{d}(\omega_\mu v^\mu) = \partial_\nu \omega_\mu v^\mu \mathrm{d}x^\nu + \omega_\mu \partial_\nu v^\mu \mathrm{d}x^\nu = (\partial_\nu \omega_\mu v^\mu + \omega_\mu \partial_\nu v^\mu)\mathrm{d}x^\nu, \tag{19}$$

$$(\mathrm{d}\boldsymbol{\omega})v = (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)v^\mu \mathrm{d}x^\nu = (\partial_\mu \omega_\nu v^\mu - \partial_\nu \omega_\mu v^\mu)v^\mu \mathrm{d}x^\nu, \tag{20}$$

从而有

$$\mathrm{d}\boldsymbol{\omega}(v) + (\mathrm{d}\boldsymbol{\omega})v = (v^\mu \partial_\mu \omega_\nu + \omega_\mu \partial_\nu v^\mu)\mathrm{d}x^\nu = \mathcal{L}\omega_\nu \mathrm{d}x^\nu = \mathcal{L}_\nu(\omega_\nu \mathrm{d}x^\nu) = \mathcal{L}_\nu \boldsymbol{\omega}. \tag{21}$$

令 i_v 为矢量场 v 与 1 形式 $\boldsymbol{\omega}$ 之间的内积, 于是就有

$$\mathrm{d}\circ i_v + i_v\circ \mathrm{d} = \mathcal{L}_v, \tag{22}$$

利用 $\mathrm{d}\circ\mathrm{d}=0$ 有

$$\mathrm{d}\circ\mathcal{L}_v = \mathrm{d}\circ i_v\circ\mathrm{d} = \mathcal{L}_v\circ\mathrm{d}. \tag{23}$$

Exercise 6 :

根据联络系数:

$$\tilde{\Gamma}^{\tau}{}_{lk}(x) = \frac{\partial\tilde{x}^\tau}{\partial x_\rho}\frac{\partial x^\mu}{\partial\tilde{x}_l}\frac{\partial x^\sigma}{\partial\tilde{x}_k}\Gamma^{\rho}{}_{\mu\sigma}(x) + \frac{\partial\tilde{x}^\tau}{\partial x_\rho}\frac{\partial^2\tilde{x}^\rho}{\partial\tilde{x}_l\partial\tilde{x}_k}$$

在不同坐标系的变换关系, 证明对 $\forall$ 协变适量场 B_μ,

$$\nabla_\lambda B_\mu \equiv \partial_\lambda B_\mu - \Gamma^{\sigma}{}_{\mu\lambda}B_\sigma$$

是 $(0,2)$ 型张量.

证明. 由于

$$\begin{aligned}
\nabla'_\lambda B'_\mu &= \partial'_\lambda B'_\mu - \Gamma'^{\nu}_{\mu\nu}B'_\nu \\
&= \left(\frac{\partial x^\rho}{\partial x'^\lambda}\right)\partial_\rho\left(\frac{\partial x^\tau}{\partial x'^\mu}B_\tau\right) - \left(\frac{\partial x'^\nu}{\partial x^\sigma}\frac{\partial x^\xi}{\partial x'^\mu}\frac{\partial x^\eta}{\partial x'^\lambda}\Gamma^\sigma_{\xi\eta} + \frac{\partial x'^\nu}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x'^\mu\partial x'^\lambda}\right)\left(\frac{\partial x^\alpha}{\partial x'^\nu}B_\alpha\right) \\
&= \frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial^2 x^\tau}{\partial x'^\mu\partial x^\rho}B_\tau + \frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial B_\tau}{\partial x^\rho} - \left(\frac{\partial x^\xi}{\partial x'^\mu}\frac{\partial x^\eta}{\partial x'^\lambda}\Gamma^\alpha_{\xi\eta} + \frac{\partial^2 x^\alpha}{\partial x'^\mu\partial x'^\lambda}\right)B_\alpha \\
&= \frac{\partial^2 x^\tau}{\partial x'^\mu\partial x'^\lambda}B_\tau - \frac{\partial^2 x^\alpha}{\partial x'^\mu\partial x'^\lambda}B_\alpha + \frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial B_\tau}{\partial x^\rho} - \frac{\partial x^\xi}{\partial x'^\mu}\frac{\partial x^\eta}{\partial x'^\lambda}\Gamma^\alpha_{\xi\eta}B_\alpha \\
&= \frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\tau}{\partial x'^\mu}\left[B_{\tau,\rho} - \Gamma^\alpha_{\tau\rho}B_\alpha\right] = \frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\tau}{\partial x'^\mu}\nabla_\tau B_\rho,
\end{aligned} \tag{24}$$

所以 $\nabla_\lambda B_\mu$ 是 $(0,2)$ 型张量. □

Exercise 7 :

(a) 设 $\gamma(t)$ 为测地线, 切矢为 t^a. 证明其重参数化 $\tilde{\gamma}(\tilde{t})$ 的切矢 $\tilde{t}^a$ 满足

$$\tilde{t}^b\nabla_b\tilde{t}^a = \alpha\tilde{t}^a$$

(其中 α 为 $\gamma(t)$ 上的某函数)

(b) 证明（非类光）测地线的线长参数必为仿射参数.

证明. **a:** 由于

$$t^a = \left(\frac{\partial}{\partial t}\right)^a = \left(\frac{\partial}{\partial \tilde{t}}\right)^a \frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} = \tilde{t}^a \frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}, \tag{25}$$

所以有

$$0 = t^b \nabla_b t^a = \tilde{t}^b \frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} \nabla_b \left(\tilde{t}^a \frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}\right) = \tilde{t}^b \left(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}\right)^2 \nabla_b \tilde{t}^a + \underbrace{\tilde{t}^b \frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} \tilde{t}^a \nabla_b \left(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}\right)}_{A}, \tag{26}$$

其中, 利用 $v^a \nabla_a f = v(f)$ 可得

$$A = \tilde{t}^a \frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} \tilde{t}^b \nabla_b \left(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}\right) = \tilde{t}^a \frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} \tilde{t}^b \left(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}\right) = \tilde{t}^a \frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} \frac{\partial}{\partial \tilde{t}} \left(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}\right) = \tilde{t}^a \frac{\mathrm{d}^2\tilde{t}}{\mathrm{d}t^2}. \tag{27}$$

从而

$$\tilde{t}^b \left(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}\right)^2 \nabla_b \tilde{t}^a + \tilde{t}^a \frac{\mathrm{d}^2\tilde{t}}{\mathrm{d}t^2} \Rightarrow \tilde{t}^b \nabla_b \tilde{t}^a = -\tilde{t}^a \frac{\mathrm{d}^2\tilde{t}}{\mathrm{d}t^2} \left(\frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}\right)^2 = \alpha \tilde{t}^a. \tag{28}$$

b: 设 $\gamma(t)$ 是以仿射参数 t 为参数的测地线, 沿着 $\gamma(t)$ 的切矢为 $T^a(t) \equiv T^a(\gamma(t))$, 则有 $T^c \nabla_c T^a = 0$ 以及 $T^2 = T^a T^b g_{ab}$. 又由于 g_{ab} 与 ∇_a 适配, 即 $\nabla_a g_{bc} = 0$, 于是就有

$$T^c \nabla_c T^2 = T^c \nabla_c (T^a T^b g_{ab})(T^c \nabla_c T^a) T^b g_{ab} + T^a (T^c \nabla_c T^b) g_{ab} + T^a T^b (T^c \nabla_c g_{ab}) = 0, \tag{29}$$

这表明以仿射参数为参数的测地线切矢长度沿线为常数, 即 $|T| = C$. 于是就有测地线线长 l 为

$$l = \int_{t_0}^{t} |T(t')| \mathrm{d}t' = C(t - t_0), \tag{30}$$

可知这一条测地线也可以用重参数化后的 $\gamma'(l)$ 描述, 其中 l 为线长参数. 根据定理 3-3-3[1], 当 $l = at + b$(a, b 为常数, 且 $a \neq 0$) 时,l 也是 $\gamma'(l)$ 的仿射参数. □

Exercise 8 :

对于无挠时空, 请证明:

(a)$R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho}$

(b)${R^\nu}_{[\mu\kappa\lambda]} = 0$

(c)$\nabla_\mu \nabla_\kappa \xi_\lambda = {R^\nu}_{\mu\kappa\lambda} \xi_\nu$, 其中 ξ_ν 为 killing 矢量场.

[1]梁灿彬, 周彬. 微分几何入门与广义相对论 (上册). 科学出版社, 2009. 第 69-70 页.

证明. **a:** 在联络系数各个分量都为零的局部惯性系中, 有

$$
\begin{aligned}
R_{\nu\mu\kappa\lambda} &= R^{\rho}{}_{\mu\kappa\lambda}g_{\rho\nu} = g_{\rho\nu}(\partial_{\kappa}\Gamma^{\rho}_{\mu\lambda} + \Gamma^{\rho}_{\sigma\kappa}\Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\sigma}_{\mu\kappa} - \partial_{\lambda}\Gamma^{\rho}_{\mu\kappa}) \\
&= g_{\rho\nu}(\partial_{\kappa}\Gamma^{\rho}_{\mu\lambda} - \partial_{\lambda}\Gamma^{\rho}_{\mu\kappa}) \\
&= \frac{1}{2}g_{\rho\nu}\partial_{k}\left[g^{\sigma\rho}(g_{\sigma\mu,\lambda} + g_{\sigma\lambda,\mu} - g_{\mu\lambda,\sigma})\right] - \frac{1}{2}g_{\rho\nu}\partial_{\lambda}\left[g^{\sigma\rho}(g_{\sigma\mu,\kappa} + g_{\sigma\kappa,\mu} - g_{\mu\kappa,\sigma})\right] \\
&= \frac{1}{2}g_{\rho\nu}g^{\sigma\rho}(g_{\sigma\mu,\lambda\kappa} + g_{\sigma\mu,\lambda\kappa} - g_{\mu\lambda,\sigma\kappa}) - \frac{1}{2}g_{\rho\nu}g^{\sigma\rho}(g_{\sigma\mu,\kappa\lambda} + g_{\sigma\kappa,\mu\lambda} - g_{\mu\kappa,\sigma\lambda}) \\
&= \frac{1}{2}(g_{\nu\lambda,\mu\kappa} + g_{\mu\kappa,\nu\lambda} - g_{\mu\lambda,\nu\kappa} - g_{\nu\kappa,\mu\lambda}),
\end{aligned}
\tag{31}
$$

从而有

$$
R_{\mu\nu\kappa\lambda} = \frac{1}{2}(g_{\mu\lambda,\nu\kappa} + g_{\nu\kappa,\mu\lambda} - g_{\nu\lambda,\mu\kappa} - g_{\mu\kappa,\nu\lambda}),
\tag{32}
$$

所以有 $R_{\mu\nu\kappa\lambda} = -R_{\nu\mu\kappa\lambda}$.

b: 在联络系数各个分量全部为 0 的局部惯性系, 有

$$
R^{\nu}{}_{\mu\kappa\lambda} = \Gamma^{\nu}_{\mu\lambda,\kappa} - \Gamma^{\nu}_{\mu\kappa,\lambda} = 2\Gamma^{\nu}_{\mu[\lambda,\kappa]},
\tag{33}
$$

从而

$$
R^{\nu}{}_{(\mu\kappa\lambda)} = 2\Gamma^{\nu}_{(\mu[\lambda,\kappa])} = 0.
\tag{34}
$$

c: 由黎曼曲率的定义[2]与 killing 方程 $\nabla_{\lambda}\xi_{\mu} + \nabla_{\mu}\xi_{\lambda} = 0$ 可得

$$
R^{\nu}{}_{\mu\kappa\lambda}\xi_{\nu} = (\nabla_{\lambda}\nabla_{\kappa} - \nabla_{\kappa}\nabla_{\lambda})\xi_{\mu} = -\nabla_{\lambda}\nabla_{\mu}\xi_{\kappa} - \nabla_{\kappa}\nabla_{\lambda}\xi_{\mu},
\tag{35}
$$

分别作两次轮换 $\mu \to \kappa \to \lambda \to \mu$, 得到

$$
R^{\nu}{}_{\kappa\lambda\mu}\xi_{\nu} = -\nabla_{\mu}\nabla_{\kappa}\xi_{\lambda} - \nabla_{\lambda}\nabla_{\mu}\xi_{\kappa},
\tag{36}
$$

和

$$
R^{\nu}{}_{\lambda\mu\kappa}\xi_{\nu} = -\nabla_{\kappa}\nabla_{\lambda}\xi_{\mu} - \nabla_{\mu}\nabla_{\kappa}\xi_{\lambda}.
\tag{37}
$$

式 35-式 36-式 37, 有

$$
2\nabla_{\mu}\nabla_{k}\xi_{\lambda} = (R^{\nu}{}_{\mu\kappa\lambda} - R^{\nu}{}_{\kappa\lambda\mu} - R^{\nu}{}_{\lambda\mu\kappa})\xi_{\nu} = 2R^{\nu}{}_{\mu\kappa\lambda}\xi_{\nu} \Rightarrow \nabla_{\mu}\nabla_{k}\xi_{\lambda} = R^{\nu}{}_{\mu\kappa\lambda}\xi_{\nu},
\tag{38}
$$

[2]黄超光. 广义相对论讲义. 科学出版社,2023. 第 97 页.

这里用到了 **b** 中的结论,

$$R^{\nu}{}_{[\mu\kappa\lambda]} = 0 \Longleftrightarrow R^{\nu}{}_{\mu\kappa\lambda} + R^{\nu}{}_{\kappa\lambda\mu} + R^{\nu}{}_{\lambda\mu\kappa} = 0. \tag{39}$$

□

Exercise 9 :

二维球面度规为：$ds^2 = d\theta^2 + sin^2(\theta)d\phi^2$, 请计算这个度规的克氏符 $\Gamma^{\rho}{}_{\mu\nu}$；黎曼曲率张量 $R^{\mu}{}_{\nu\lambda\rho}$；里奇张量 $R_{\mu\nu}$ 和曲率标量 R.

解: 克氏符: 度规非零分量仅有 $g_{\theta\theta} = 1, g_{\phi\phi} = \sin^2\theta$, 且有 $g^{\theta\theta} = 1, g^{\phi\phi} = \sin^{-2}\theta$. 根据

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}), \tag{40}$$

可得到非零克氏符仅有 $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$.

黎曼曲率: 根据

$$R^{\nu}{}_{\mu\kappa\lambda} = \Gamma^{\nu}_{\mu\lambda,\kappa} - \Gamma^{\nu}_{\mu\kappa,\lambda} + \Gamma^{\nu}_{\sigma\kappa}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\nu}_{\sigma\lambda}\Gamma^{\sigma}_{\mu\kappa}, \tag{41}$$

得非零的黎曼曲率为 $R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta = -R^{\theta}{}_{\phi\phi\theta}, R^{\phi}{}_{\theta\theta\phi} = -1 = -R^{\phi}{}_{\theta\phi\theta}$.

里奇张量: 根据 $R_{\mu\lambda} = R^{\nu}{}_{\mu\nu\lambda}$ 得到 $R_{\phi\phi} = \sin^2\theta, R_{\theta\theta} = 1$.

曲率标量: 根据 $R = g^{\mu\lambda}R_{\mu\lambda}$ 得到 $R = 2$.

Exercise 10 :

对于二维闵氏时空：$ds^2 = -dt^2 + dx^2$, 给出所有独立的 Killing 矢量场.

解: 二维闵氏度规为

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} \dfrac{\partial\eta_{\mu\nu}}{\partial t} = 0, \\ \dfrac{\partial\eta_{\mu\nu}}{\partial x} = 0, \end{cases} \tag{42}$$

于是两个显然的 killing 矢量为 $\xi^a_1 = \left(\frac{\partial}{\partial t}\right)^a, \xi^a_2 = \left(\frac{\partial}{\partial x}\right)^a$. 第三个 killing 矢量的求解有下面两种办法:

利用 killing 方程

由于二维闵氏空间不存在非零的克氏符, 所以有

$$\begin{cases} \xi_{t;t} = 0, \\ \xi_{t;x} + \xi_{x;t} = 0 \\ \xi_{x;x} = 0 \end{cases} \Rightarrow \begin{cases} \xi_{t,t} = \eta_{tt}\xi^t_{,t} = 0, \\ \xi_{t,x} + \xi_{x;t} = \eta_{tt}\xi^t_{,x} + \eta_{xx}\xi^x_{,t} = 0 \\ \xi_{x,x} = \eta_{xx}\xi^x_{,x} = 0 \end{cases} \tag{43}$$

即

$$\begin{cases} \dfrac{\partial \xi^t}{\partial t} = 0 \\ -\dfrac{\partial \xi^t}{\partial x} + \dfrac{\partial \xi^x}{\partial t} = 0 \\ \dfrac{\partial \xi^x}{\partial x} = 0 \end{cases} \Rightarrow \begin{cases} \xi^t = bx + a^t \\ \xi^x = bt + a^x \end{cases} \tag{44}$$

其中 b, a^t, a^x 为三个积分常数, 给出三个线性独立的解为 $(1,0), (0,1), (x,t)$. 显然前两个对应于 killing 矢量 ξ_1^a, ξ_2^a, 第三个对应于 killing 矢量 $\xi_3^a = x\left(\dfrac{\partial}{\partial t}\right)^a + t\left(\dfrac{\partial}{\partial x}\right)^a$.

利用坐标变换

定义新坐标如下所示

$$\begin{cases} x = \psi \cosh \eta \\ t = \psi \sinh \eta \end{cases} \tag{45}$$

其中 $0 < \psi < \infty, -\infty < \eta < \infty$. 于是二维闵氏线元可以改写为

$$\mathrm{d}s^2 = \mathrm{d}\psi^2 - \psi^2 \mathrm{d}\eta^2, \tag{46}$$

可见线元表达式中不含 η, 则 $\left(\dfrac{\partial}{\partial \eta}\right)^a = t\left(\dfrac{\partial}{\partial x}\right)^a + x\left(\dfrac{\partial}{\partial t}\right)^a$ 即为待求的 killing 矢量.

Exercise 11 :

在坐标变换 $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x)$ 和弱场近似下

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), |h_{\mu\nu}| \ll 1, |\partial_\rho \cdots \partial_\sigma h_{\mu\nu}| \ll 1,$$

求

(a)$h'_{\mu\nu}(x')$ 和 $h_{\mu\nu(x)}$ 的变换关系.

(b) 写出黎曼张量 $R_{\mu\nu\rho\sigma}(x)$ 与 $h_{\mu\nu}(x)$ 的关系式.

(c) 证明 $R'_{\mu\nu\rho\sigma}(x') = R_{\mu\nu\rho\sigma}(x)$.

(d) 写出在 $\partial^\mu h_{\mu\nu} = 0$ 的规范下 $h_{\mu\nu}$ 满足的运动方程.

解:(a) 由于 $x^{\mu'} = x^\mu + \varepsilon^\mu(x^\nu)$, 所以

$$\Lambda^{\mu'}{}_\nu = \frac{\partial x^{\mu'}}{\partial x^\nu} = \delta^\mu{}_\nu + \varepsilon^\mu{}_\nu \Rightarrow \Lambda^\nu{}_{\mu'} = (\Lambda^{\mu'}{}_\nu)^{-1} = (\delta^\mu{}_\nu + \varepsilon^\mu{}_\nu)^{-1} = \delta^\mu{}_\nu - \varepsilon^\mu{}_\nu, \tag{47}$$

从而

$$\begin{aligned} g_{\mu'\nu'} &= \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}g_{\mu\nu} \\ &= (1-\varepsilon^{\mu}{}_{,\mu})(1-\varepsilon^{\nu}{}_{,\nu})(\eta_{\mu\nu}+h_{\mu\nu}) \\ &\approx \eta_{\mu\nu}+h_{\mu\nu}-\varepsilon_{\mu,\nu}-\varepsilon_{\nu,\,\mu}. \end{aligned} \tag{48}$$

又因为 $g_{\mu'\nu'} = \eta_{\mu'\nu'} + h_{\mu'\nu'} = \eta_{\mu\nu} + h'_{\mu\nu}$, 所以有

$$h'_{\mu\nu} = h_{\mu\nu} - \varepsilon_{\mu,\nu} - \varepsilon_{\nu,\mu}. \tag{49}$$

(b): 因为

$$\begin{aligned} R^{\sigma}{}_{\beta\mu\nu} &= \partial_\mu\Gamma^{\sigma}{}_{\beta\nu} - \partial_\nu\Gamma^{\sigma}{}_{\beta\mu} + \Gamma^{\sigma}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\beta\nu} - \Gamma^{\sigma}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\beta\mu} \\ &\approx \partial_\mu\Gamma^{\sigma}{}_{\beta\nu} - \partial_\nu\Gamma^{\sigma}{}_{\beta\mu} \\ &= \frac{1}{2}g^{\sigma\tau}[\partial_\mu(g_{\tau\beta,\nu}) + g_{\tau\nu,\beta} - g_{\beta\nu,\tau} - \partial_\nu(g_{\tau\beta,\mu} + g_{\tau\mu,\beta} - g_{\beta\mu,\tau})] \\ &\approx \frac{1}{2}\eta^{\sigma\tau}(h_{\tau\beta,\nu\mu} + h_{\tau\nu,\beta\mu} - h_{\beta\mu,\tau\nu} - h_{\tau\beta,\mu\nu} - h_{\tau\mu,\beta\nu} + h_{\beta\mu,\tau\nu}) \\ &= \frac{1}{2}\eta^{\sigma\tau}(+h_{\tau\nu,\beta\mu} - h_{\beta\mu,\tau\nu} - h_{\tau\mu,\beta\nu} + h_{\beta\mu,\tau\nu}), \end{aligned} \tag{50}$$

所以

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= g_{\alpha\sigma}R^{\sigma}{}_{\beta\mu\nu} \\ &\approx \frac{1}{2}\eta_{\alpha\sigma}\eta^{\sigma\tau}(+h_{\tau\nu,\beta\mu} - h_{\beta\mu,\tau\nu} - h_{\tau\mu,\beta\nu} + h_{\beta\mu,\tau\nu}) \\ &= \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu} - h_{\alpha\mu,\beta\nu}). \end{aligned} \tag{51}$$

(c): 由于 $h'_{\mu\nu} = h_{\mu\nu} - \varepsilon_{\mu,\nu} - \varepsilon_{\nu,\mu}$, 而 $|\partial_\rho \cdots \partial_\sigma \varepsilon^{n}_{\nu}| \ll 1$, 所以

$$\begin{cases} h'_{\alpha\nu,\beta\mu} = h_{\alpha\nu,\beta\mu}, \\ h'_{\beta\mu,\alpha\nu} = h_{\beta\mu,\alpha\nu}, \\ h'_{\beta\nu,\alpha\mu} = h_{\beta\nu,\alpha\mu}, \\ h'_{\alpha\mu,\beta\nu} =_{\alpha\mu,\beta\nu} \end{cases} \Rightarrow R'_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}. \tag{52}$$

(d): 由于

$$\begin{aligned} R_{\mu\nu} &\approx \partial_\lambda\Gamma^{\lambda}_{\mu\nu} - \partial_\nu\Gamma^{\lambda}_{\mu\nu} \\ &\approx \frac{1}{2}\eta^{\lambda\tau}(h_{\tau\nu,\mu\lambda} + h_{\mu\lambda,\tau\nu} - h_{\mu\nu,\tau\lambda} - h_{\tau\lambda,\mu\nu}) \\ &= \frac{1}{2}(\eta^{\lambda\tau}h_{\tau\nu,\mu\lambda} + \eta^{\lambda\tau}h_{\mu\lambda,\tau\nu} - \Box h_{\mu\nu} - \eta^{\lambda\tau}h_{\tau\lambda,\mu\nu}) \\ &= \frac{1}{2}(h_{\nu}{}^{\lambda}{}_{,\lambda\mu} + h_{\mu}{}^{\lambda}{}_{,\lambda\nu} - \Box h_{\mu\nu} - h_{,\mu\nu}), \end{aligned} \tag{53}$$

或者写为

$$R_{\alpha\beta}=\frac{1}{2}\left(h_{\beta\mu,\alpha}{}^{\mu}+h_{\alpha\mu,\beta}{}^{\mu}-h_{\alpha\beta}-h_{\alpha\beta,\mu}{}^{\mu}\right), \tag{54}$$

于是里奇标量为

$$\begin{aligned} R &\approx \eta^{\mu\nu}\frac{1}{2}\left(h_{\nu\sigma,\mu}{}^{f}+\underline{h_{\mu\sigma,v'}}-h_{\mu\nu}-h_{\mu\nu,\sigma}\right) \\ &=\frac{1}{2}\left(h_{v\sigma}{}^{,v\sigma}-h_{,v}{}^{,v}-h_{\rho\sigma}{}^{,\sigma}\right)+\frac{1}{2}\eta^{\mu v}\underline{h_{\mu\sigma,v}}{}^{\sigma} \\ &=\frac{1}{2}\left(h_{v\sigma}{}^{v\sigma}-h_{,\nu}{}^{,\nu}-h_{\rho}{}^{\sigma}\right)+\frac{1}{2}h^{\mu\sigma}_{\mu\sigma} \\ &=h_{v\sigma}{}^{v\sigma}-h_{\mu}{}^{\mu\nu}. \end{aligned} \tag{55}$$

从而

$$\begin{aligned} G_{\alpha\beta} &=\frac{1}{2}\left(h^{\mu}_{\beta\mu,\alpha}+h^{\mu}_{\alpha\mu,\beta}-h^{\mu}_{\alpha\beta,\mu}-h_{\alpha\beta}\right)-\frac{1}{2}\eta_{\alpha\beta}\left[h_{vo}{}^{\nu\sigma}-h_{\mu}{}^{\mu}\right] \\ &=\frac{1}{2}\left(\bar{h}^{\mu}_{\beta\mu\alpha}+\bar{h}^{\mu}_{\alpha\mu,\beta}-\bar{h}^{\mu}_{\alpha\beta,\mu}+\underline{\bar{h}_{\alpha\beta}}-\frac{1}{2}\left\{\underline{\eta_{\beta\mu}\bar{h}^{\mu}_{\alpha}}+\underline{\eta_{\alpha\mu}\bar{h}^{\mu}_{\beta}}-\overline{\eta_{\alpha\beta}\bar{h}^{\mu}_{\alpha}}\right\}\right) \\ &-\frac{1}{2}\eta_{\alpha\beta\beta}\left[\bar{h}_{vo}{}^{vv\sigma}-\overline{\frac{1}{2}\eta_{v\sigma}\bar{h}^{v\sigma}}+\overline{\bar{h}_{\mu}{}^{\mu}}\right] \\ &=-\frac{1}{2}\left[\bar{h}^{\mu}_{\alpha\beta,\mu}+\eta_{\alpha\beta}\bar{h}^{\mu\nu}_{\mu\nu}-\bar{h}^{\mu}_{\alpha\mu,\beta}-\bar{h}^{\mu}_{\beta\mu,\alpha}\right] \\ &\xlongequal{\bar{h}^{\mu\nu}{}_{,\nu}=0}-\frac{1}{2}\bar{h}^{\mu}_{\alpha\beta,\mu}=8\pi GT_{\mu\nu}, \end{aligned} \tag{56}$$

即

$$\Box\bar{h}_{\alpha\beta}=-16\pi GT_{\alpha\beta}. \tag{57}$$

Exercise 12 :

若将 $R_g=2GM$ 称为质量为 M 的黑洞的引力半径, 在 $r=10R_g$ 处的观察者收到发自静止于 $r=5R_g$ 处的光源发出的光. 试求接收到的光的频率和发出时的频率的比值 (r 为径向坐标).

解:

$$\frac{\omega_o}{\omega_s}=\sqrt{\frac{1-\dfrac{2Gm}{r_s}}{1-\dfrac{2Gm}{r_o}}}\xlongequal[r_s=5R_g]{r_o=10R_g}\sqrt{\frac{1-\dfrac{2Gm}{5R_g}}{1-\dfrac{Gm}{5R_g}}}\xlongequal{R_g=2GM}\frac{2\sqrt{2}}{3}. \tag{58}$$

Exercise 13 :

对于施瓦西黑洞:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

其中,

$$f(r) = 1 - \frac{2M}{r}.$$

计算其总质量, 并验证对任意给定的视界外的体积 $V(r = Constant)$, 得到的结果相同。

解: 设 $n^\mu = (f(r)^{-1/2}, 0, 0, 0), \sigma^\mu = (0, f(r)^{1/2}, 0, 0)$ 分别为超曲面 Σ 与 $\partial\Sigma$ 的单位法矢量, 则有 $n_\mu = (-f(r)^{1/2}, 0, 0, 0), \sigma_\mu = (0, f(r)^{-1/2}, 0, 0)$，所以有

$$\begin{aligned} n_\mu\sigma_\nu\nabla^\mu\xi^\nu &= -\nabla^t\xi^r = -g^{tt}\nabla_t\xi^r = -g^{tt}(\partial_t\xi^r + \Gamma^t_{t\mu}\xi^\mu) \\ &= -g^{tt}\Gamma^r_{tt} = \left(1 - \frac{2M}{r}\right)^{-1}\frac{M}{r^2}\left(1 - \frac{2M}{r}\right) \\ &= \frac{M}{r^2}, \end{aligned} \tag{59}$$

从而史瓦西黑洞的 Komar 质量为

$$\begin{aligned} M_k &= \frac{1}{4\pi}\int_{\partial\Sigma}\mathrm{d}An_\mu\sigma_\nu\nabla^\mu\xi^\nu \\ &= \frac{1}{4\pi}\int\mathrm{d}\theta\mathrm{d}\phi r^2\sin\theta\frac{M}{r^2} \\ &= \frac{M}{4\pi}\int_0^\pi\sin\theta\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi = M. \end{aligned} \tag{60}$$

上述计算并不依赖于 r, 因此对任意给定的视界外的体积 $V(r = Constant)$, 得到的结果相同.

Exercise 14 :

对于 AdS 时空, 证明未来类时测地线在有限固有时内无法到达边界 ($\mathcal{J}$) 。(注：考虑静态坐标系中沿径向运动的粒子)

证明. Ads 时空沿着径向的线元为

$$(\mathrm{d}s^*)^2 = -\left(1 + l^{-2}r^2\right)\mathrm{d}t^2 + \left(1 + l^{-2}r^2\right)^{-1}\mathrm{d}r^2, \tag{61}$$

类时测地线要求

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = -\left(1 + r^{-2}r^2\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1 + l^{-2}r^2\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}r}\right)^2 = -1. \tag{62}$$

所以有守恒量

$$E=-g_{\mu v}\left(\frac{\partial}{\partial t}\right)^{\mu}\left(\frac{\partial}{\partial r}\right)^{v}=-\left[-\left(1+l^{-2}r^{2}\right)\right]\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)=\left(1+l^{-2}r^{2}\right)\frac{\mathrm{d}t}{\mathrm{d}r}, \tag{63}$$

代入式 62, 得到如下的微分方程

$$\left(\frac{\mathrm{d}r}{\mathrm{d}r}\right)^{2}+l^{-2}r^{2}=E^{2}-1, \tag{64}$$

其通解为

$$r(\tau)=\frac{l\sqrt{E^{2}-1}|\tan(\tau/l)|}{\sqrt{1+|\tan(\tau/l)]^{2}}}=l\sqrt{E^{2}-1}|\sin(\tau/C)|\xlongequal{\tau\to\infty}\leq l\sqrt{E^{2}-1}|, \tag{65}$$

其中 C 为积分常数. 可见, 径向坐标 $r(\tau)$ 在 $\tau\to\infty$ 时是有界的, 这就证明了未来类时测地线在有限固有时内无法到达边界. □