一份 (不怎么严谨的) 高等量子力学习题解答

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Problem 1

试阐述本课程所讲授的狄拉克量子力学四大公理与三大数学操作规则, 其中每项必须给出两种情形下的关键公式 (各 6 分, 共 42 分)

- **量子力学四大公理**

1. 重叠原理：任何量子态都是其他两个或者多个量子态重叠后产生的; 任何两个或者多个量子态叠加起来的结果产生一个新的量子态. 即 $\forall c_1, c_2 \in \mathbb{C}$ 有

$$c_1 |A\rangle + c_2 |B\rangle = |C\rangle, \tag{1}$$

$$c_1 \langle A| + c_2 \langle B| = \langle C|, \tag{2}$$

$$\langle A|B\rangle = \langle B|A\rangle^*, \langle A|A\rangle \geq 0. \tag{3}$$

2. 测量原理：量子系统的每次测量值必定为厄米算子的本征值, 相应的本征矢量对应的量子态就是测量时和测量后量子体系的本征态. 对算子 $\hat{\alpha}$, 设相应的本征值为 α, 本征矢为 $|\alpha\rangle$, 则有本征方程:

$$\hat{\alpha} |\alpha\rangle = \alpha |\alpha\rangle. \tag{4}$$

厄米算子的本征值为实数, 而且厄米算子对应于不同本征值的本征矢量彼此正交. 设厄米算子的两个本征值为 h_1, h_2, 且 $h_1 \neq h_2$, 则 $\langle h_2 \mid h_1\rangle = 0$.

3. 狄拉克正则量子化条件:

$$[\hat{x}, \hat{p}] = i\hbar. \tag{5}$$

不确定关系: 对于两个厄米算子 (即可观测量算子)$\hat{u}$ 和 $\hat{v}$, 存在不等式

$$\left\langle (\Delta\hat{u})^2 \right\rangle \left\langle (\Delta\hat{v})^2 \right\rangle \geq \frac{1}{4} |\langle [\hat{u}, \hat{v}]\rangle|^2 \Rightarrow \Delta x \Delta p \geq \frac{\hbar}{2} \tag{6}$$

4. 薛定谔方程:

$$i\hbar \frac{\partial}{\partial t} \Psi(r,t) = \hat{H} \Psi(r,t), \tag{7}$$

海森堡运动方程

$$i\hbar \frac{\mathrm{d}\hat{A}(t)}{\mathrm{d}t} = [\hat{A}(t), \hat{H}]. \tag{8}$$

- **量子力学三大数学操作**

1. 表象 (表示): 用厄米算子 $\hat{\boldsymbol{\alpha}}$ 的本征矢量集 $\{|\alpha_i\rangle\}$ 做量子态所对应的矢量空间的基矢组, 把抽象的矢量、算子与操作都用数字表达出来, 叫作矢量和算子的一种 $\boldsymbol{\alpha}$ 表示. 对于离散谱有正交归一性和完备性关系:

$$\begin{aligned} \langle \alpha_i \mid \alpha_j \rangle &= \delta_{ij}, \\ \sum_i |\alpha_i\rangle \langle \alpha_i| &= 1. \end{aligned} \tag{9}$$

对于连续谱则有

$$\begin{aligned}\langle\alpha(x)\,|\alpha(y)\rangle &= \delta(x-y),\\ \int \mathrm{d}x\,|\alpha(x)\rangle\,\langle\alpha(x)| &= 1.\end{aligned} \tag{10}$$

右矢量、左矢量、内积和线性算子的表示为

$$\begin{cases} |A\rangle = \sum_{n=1}^{\infty}|n\rangle\,\langle n \mid A\rangle = \sum_{n=1}^{\infty} a_n|n\rangle,\\ \langle A| = \sum_{n=1}^{\infty}\langle A \mid n\rangle\langle n| = \sum_{n=1}^{\infty} a_n^*\langle n|,\\ \langle A \mid B\rangle = \sum_{n=1}^{\infty}\langle A \mid n\rangle\langle n \mid B\rangle = \sum_{n=1}^{\infty} a_n^* b_n,\\ \hat{\alpha} = \sum_{n,m}|n\rangle\langle n|\hat{\alpha}|m\rangle\langle m|.\end{cases} \tag{11}$$

2. 自旋沿任意方向的的变换

$$\widehat{U}\,|\hat{\sigma}_z,+\rangle = |\hat{\sigma}_n,+\rangle\,, \widehat{U}\,|\hat{\sigma}_z,-\rangle = |\hat{\sigma}_n,-\rangle\,, \tag{12}$$

其中 $\widehat{U}(\vec{\mathbf{n}},\phi) = \exp\left(-\frac{\hat{\vec{\sigma}}\cdot\vec{\mathbf{n}}\phi}{2}\right)$.

幺正表换:

$$\widehat{U}\widehat{S}_{\mathrm{Z}}\widehat{U}^{\dagger} = \hat{\vec{\mathbf{S}}}\cdot\vec{\mathbf{n}}. \tag{13}$$

3. 设线性算子 $\widehat{T}(t,t_0) = \exp\left(\frac{\hat{H}(t-t_0)}{i\hbar}\right)$, 则有薛定谔影像:

$$\begin{aligned}|A,t\rangle_S &= \widehat{T}\,(t,t_0)\,|A,t_0\rangle\,,\\ \hat{\alpha}_S(t) &= \hat{\alpha}(t_0).\end{aligned} \tag{14}$$

海森堡影像:

$$\begin{aligned}|A,t\rangle_H &= \widehat{T}^{-1}\,(t,t_0)\,|A,t\rangle_S = |A,t_0\rangle = |A\rangle\,,\\ \widehat{\alpha}_H(t) &= \widehat{T}^{-1}\,(t,t_0)\,\widehat{\alpha}_S(t)\widehat{T}\,(t,t_0) = \widehat{T}^{-1}\,(t,t_0)\,\widehat{\alpha}(t)\widehat{T}\,(t,t_0)\,.\end{aligned} \tag{15}$$

Problem 2

试采用狄拉克梯子算子推导论证出决定谐振子基态的方程式 (10 分) 并求出谐振子基态波函数 (5 分)

狄拉克升降算子的定义为:

$$\hat{a} = \frac{1}{\sqrt{2}}\left[\frac{\hat{p}}{\sqrt{m\hbar\omega}} - i\sqrt{\frac{m\omega}{\hbar}}\hat{x}\right], \hat{a}^{\dagger} = \frac{1}{\sqrt{2}}\left[\frac{\hat{p}}{\sqrt{m\hbar\omega}} + i\sqrt{\frac{m\omega}{\hbar}}\hat{x}\right], \tag{16}$$

满足如下的对易关系:

$$\left[\hat{a},\hat{a}^{\dagger}\right] = \hat{I}, [\hat{a},\hat{a}] = \left[\hat{a}^{\dagger},\hat{a}^{\dagger}\right] = 0. \tag{17}$$

于是谐振子的哈密顿量可以表示为

$$\hat{H} = \left[\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right]\hbar\omega = \left[\hat{N} + \frac{1}{2}\right]\hbar\omega, \tag{18}$$

其中 $\hat{N}=\hat{a}^{\dagger}\hat{a}$ 为厄米算符, 满足 $[\hat{N},\hat{a}]=-\hat{a},\left[\hat{N},\hat{a}^{\dagger}\right]=\hat{a}^{\dagger}$. 设 $\hat{N}|n\rangle=n|n\rangle,\langle n\mid n\rangle=1$, 有

$$n=n\langle n\mid n\rangle=\langle n|\hat{N}|n\rangle=\langle n|\hat{a}^{\dagger}\hat{a}|n\rangle=|\hat{a}|n\Big\rangle\,|^{2}\Rightarrow n\geq 0. \tag{19}$$

一维量子力学体系的束缚态能级不简并. 所以以上两式的成立意味着:

$$\hat{a}|n\rangle=\lambda(n)|n-1\rangle,\hat{a}^{\dagger}|n\rangle=\nu(n)|n+1\rangle. \tag{20}$$

设 $\hat{N}$ 的本征态矢均满足归一化条件，如此即有:

$$\begin{aligned}
|\lambda(n)|^{2}&=[\lambda(n)^{*}\langle n-1|]\cdot[\lambda(n)|n-1\rangle]\\
&=\langle n|\hat{a}^{\dagger}a|n\rangle\\
&=\langle n|\hat{N}|n\rangle\\
&=n\Rightarrow\lambda(n)=\sqrt{n},\\
|\nu(n)|^{2}&=[\nu(n)^{*}\langle n+1|]\cdot[\nu(n)|n+1\rangle]\\
&=\langle n|a\hat{a}^{\dagger}|n\rangle\\
&=\langle n|(\hat{N}+\hat{I})|n\rangle\\
&=n+1\Rightarrow\nu(n)=\sqrt{n+1}.
\end{aligned} \tag{21}$$

所以,

$$\hat{a}|n\rangle=\sqrt{n}|n-1\rangle,\hat{a}^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle. \tag{22}$$

由于 $\hat{N}$ 的本征值非负, 上述通过降算符作用获取 $\hat{N}$ 属于较低本征值的本征态矢量的过程不可能无限制的持续下去. $\hat{N}$ 必定存在着最小的本征值 n_G, 其对应的本征态 $|n_G\rangle$ 使得降算符的作用终止:

$$\hat{a}\,|n_G\rangle=0\Rightarrow n_G=0. \tag{23}$$

$$\hat{a}\,|0\rangle=0\Rightarrow 0=\langle x|\hat{a}|0\rangle=\frac{1}{\sqrt{2}}\langle x|\left[\frac{\hat{p}}{\sqrt{m\hbar\omega}}-i\sqrt{\frac{m\omega}{\hbar}}\hat{x}\right]|0\rangle, \tag{24}$$

利用位置算符本征矢量系的完备性关系

$$\int_{-\infty}^{+\infty}dx|x\rangle\langle x|=1, \tag{25}$$

可以把基态波函数满足的方程重新表达为:

$$\begin{aligned}
0&=\langle x|\left[\frac{\hat{p}}{\sqrt{m\hbar\omega}}-i\sqrt{\frac{m\omega}{\hbar}}\hat{x}\right]|0\rangle\\
&=\int_{-\infty}^{+\infty}dy\langle x|\left[\frac{\hat{p}}{\sqrt{m\hbar\omega}}-i\sqrt{\frac{m\omega}{\hbar}}\hat{x}\right]|y\rangle\langle y\mid 0\rangle\\
&=\frac{1}{\sqrt{m\hbar\omega}}\int_{-\infty}^{+\infty}dy\langle x|\hat{p}|y\rangle\psi_0(y)-i\sqrt{\frac{m\omega}{\hbar}}\int_{-\infty}^{+\infty}dy\langle x|\hat{x}|y\rangle\psi_0(y),
\end{aligned} \tag{26}$$

因为

$$\langle x|\hat{p}|y\rangle=-i\hbar\frac{\partial}{\partial x}\delta(x-y),\quad\langle x|\hat{x}|y\rangle=x\delta(x-y), \tag{27}$$

所以上述方程最终化为如下一阶常微分方程:

$$\psi_0'(x)+\alpha^{2}x\psi_0(x)=0,\alpha=\sqrt{\frac{m\omega}{\hbar}}, \tag{28}$$

其解为:

$$\psi_0(x) = \mathcal{N} e^{-\frac{1}{2}\alpha^2 x^2}, \tag{29}$$

基态波函数的归一化条件

$$1 = \int_{-\infty}^{+\infty} dx \, \|\psi_0(x)\|^2 = \|\mathcal{N}\|^2 \int_{-\infty}^{+\infty} dx e^{-\alpha^2 x^2} = \|\mathcal{N}\|^2 \frac{\sqrt{\pi}}{\alpha}, \tag{30}$$

允许我们把归一化常数 $\mathcal{N}$ 选择为:

$$\mathcal{N} = \sqrt{\frac{\alpha}{\sqrt{\pi}}} \Rightarrow \psi_0(x) = \sqrt{\frac{\alpha}{\sqrt{\pi}}} e^{-\frac{1}{2}\alpha^2 x^2}. \tag{31}$$

Problem 3

试求无外场狄拉克方程在任意动量方向的旋量平面波解.(15 分)

狄拉克方程式如下:

$$i\hbar \frac{\partial \psi}{\partial t} = \left[c \begin{pmatrix} 0 & \hat{\sigma} \\ \hat{\sigma} & 0 \end{pmatrix} \cdot \left(\frac{\hbar}{i} \hat{\vec{\nabla}} \right) + \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix} m_0 c^2 \right] \psi, \tag{32}$$

原式中均为 4×4 的分块矩阵，将其合并得到:

$$\begin{pmatrix} m_0 c^2 & c\hat{\sigma} \cdot \vec{\mathbf{p}} \\ c\hat{\sigma} \cdot \vec{\mathbf{p}} & -m_0 c^2 \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = E \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{33}$$

能量本征值的表达式为

$$\det \begin{pmatrix} m_0 c^2 - E & c\vec{\sigma} \cdot \vec{\mathbf{p}} \\ c\vec{\sigma} \cdot \vec{\mathbf{p}} & -m_0 c^2 - E \end{pmatrix} = E^2 - m_0^2 c^4 - (c\vec{\sigma} \cdot \vec{p})^2 = 2(E^2 - 2m_0^2 c^4 - 2c^2 p^2)^2 = 0, \tag{34}$$

即 $E^2 = c^2 p^2 + m_0^2 c^4$. 代入久期方程, 先得出正能解:

$$\begin{pmatrix} m_0 c^2 & \sigma c p \\ \sigma c p & -m_0 c^2 \end{pmatrix} \begin{pmatrix} a_\sigma^+ \\ b_\sigma^+ \end{pmatrix} = E_p \begin{pmatrix} a_\sigma^+ \\ b_\sigma^+ \end{pmatrix}, \tag{35}$$

得 $b_\sigma^+ = \sigma a_\sigma^+ \dfrac{cp}{E_p + m_0 c^2}$. 由归一化条件:

$$(\varphi_\sigma^{+*}, \chi_\sigma^{+*}) \begin{pmatrix} \varphi_\sigma^+ \\ \chi_\sigma^+ \end{pmatrix} = \left(\langle \vec{\mathbf{n}}, \sigma | a_\sigma^{+*}, \langle \vec{\mathbf{n}}, \sigma | b_\sigma^{+*} \right) \begin{pmatrix} a_\sigma^+ | \vec{\mathbf{n}}, \sigma \rangle \\ b_\sigma^+ | \vec{\mathbf{n}}, \sigma \rangle \end{pmatrix} \tag{36}$$

$$= |a_\sigma^+|^2 + |b_\sigma^+|^2 = 1, \tag{37}$$

得到决定本征态的分块系数公式:

$$|a_\sigma^+|^2 + \left(\sigma a_\sigma^+ \frac{cp}{E_p + m_0 c^2} \right)^2 = 1, \tag{38}$$

取 $a_\sigma^+ = \sqrt{\frac{E_p + m_0 c^2}{2E_p}}$, 可得正能解和负能解分别为:

$$\begin{pmatrix} \varphi_\sigma^+ \\ \chi_\sigma^+ \end{pmatrix} = \sqrt{\frac{E_p + m_0 c^2}{2E_p}} \begin{pmatrix} |\vec{\mathbf{n}}, \sigma\rangle \\ \dfrac{\sigma c p}{E_p + m_0 c^2} |\vec{\mathbf{n}}, \sigma\rangle \end{pmatrix}, \tag{39}$$

$$\begin{pmatrix} \varphi_\sigma^- \\ \chi_\sigma^- \end{pmatrix} = \sqrt{\frac{E_p + m_0 c^2}{2E_p}} \begin{pmatrix} \dfrac{-\sigma c p}{E_p + m_0 c^2} |\vec{\mathbf{n}}, \sigma\rangle \\ |\vec{\mathbf{n}}, \sigma\rangle \end{pmatrix}. \tag{40}$$

Problem 4

试计算弹性散射问题中所用的三维零级推迟格林函数（8 分）并由零级定态的正交归一性导出推迟散射态的正交归一关系（7 分）.

(1) 对于三维零级推迟格林函数的计算如下:

$$\begin{aligned}
G_0(\vec{\mathbf{x}},\vec{\mathbf{x}}';z) = &\quad (2\pi\hbar)^{-3}\int d\vec{\mathbf{p}}\quad \frac{e^{i\vec{\mathbf{p}}\cdot(\vec{\mathbf{x}}-\vec{\mathbf{x}}')/\hbar}}{-|E|-\frac{p^2}{2m}} \\
&= -\frac{2m}{(2\pi\hbar)^3}\int_0^\infty p^2dp\int_0^{2\pi}d\varphi\int_0^\pi \sin\theta d\theta\frac{e^{ip|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|\cos\theta/\vec{\mathbf{x}}}}{p^2+p_0^2} \\
&= \quad\frac{2m2\pi}{(2\pi\hbar)^3}\frac{\hbar}{i|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}\int_0^\infty pdp\frac{e^{-ip|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|/\vec{\mathbf{x}}}-e^{ip|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|/\vec{\mathbf{x}}}}{p^2+p_0^2} \\
&= -\frac{2m2\pi}{(2\pi\hbar)^3}\frac{\hbar}{i|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}\int_{-\infty}^\infty pdp\frac{e^{ip|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|/\hbar}}{p^2+p_0^2} \\
&= -\frac{2m2\pi}{(2\pi\hbar)^3}\frac{\hbar}{i|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}2\pi i\left[\frac{pe^{ip|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|/\hbar}}{p^2+p_0^2}(p-ip_0)\right]_{p=ip_0} \\
&= -\frac{m}{2\pi\hbar^2}\frac{e^{-p_0|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|/\hbar}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}
\end{aligned} \tag{41}$$

(2) 对右矢量进行展开得:

$$\begin{aligned}
|\psi_a^{(\pm)}\rangle &= \frac{1}{E_a-\hat{H}\pm i\eta}\left(E_a-\hat{H}_0-\hat{V}\pm i\eta+\hat{V}\right)|\phi_a\rangle \\
&= |\phi_a\rangle+\frac{1}{E_a-\hat{H}\pm i\eta}\hat{V}|\phi_a\rangle \\
&= |\phi_a\rangle+\hat{G}^{(\pm)}(E_a)\hat{V}|\phi_a\rangle
\end{aligned} \tag{42}$$

代入两个零级态矢量计算正交性得:

$$\begin{aligned}
&\langle\psi_a^{(+)}|\psi_b^{(+)}\rangle \\
&= \langle\psi_a^{(+)}|\left[|\phi_b\rangle+\hat{G}^{(+)}(E_b)\hat{V}|\phi_b\rangle\right] \\
&= \langle\psi_a^{(+)}|\phi_b\rangle+\langle\psi_a^{(+)}|\hat{G}^{(+)}(E_b)\hat{V}|\phi_b\rangle
\end{aligned} \tag{43}$$

第一项中左矢量的展开方程为:

$$\begin{aligned}
\langle\psi_a^{(\pm)}| &= \langle\phi_a|+\langle\psi_a^{(\pm)}|\hat{V}^\dagger\left[\hat{G}_0^{(\pm)}(E_a)\right]^\dagger \\
&= \langle\phi_a|+\langle\psi_a^{(\pm)}|\hat{V}\hat{G}_0^{(\mp)}(E_a)
\end{aligned} \tag{44}$$

代入计算得:

$$\begin{aligned}
&\langle\psi_a^{(+)}|\psi_b^{(+)}\rangle \\
&= \langle\psi_a^{(+)}|\left[|\phi_b\rangle+\hat{G}^{(+)}(E_b)\hat{V}|\phi_b\rangle\right] \\
&= \langle\phi_a|\phi_b\rangle+\langle\psi_a^{(\pm)}|\hat{V}\hat{G}_0^{(-)}(E_a)|\phi_b\rangle+\langle\psi_a^{(\pm)}|\hat{G}^{(+)}(E_b)\hat{V}|\phi_b\rangle \\
&= \langle\phi_a|\phi_b\rangle+\left(\frac{1}{E_a-E_b-i\eta}+\frac{1}{E_b-E_a+i\eta}\right)\langle\psi_a^{(\pm)}|\hat{V}|\phi_b\rangle \\
&= \langle\phi_a|\phi_b\rangle
\end{aligned} \tag{45}$$

Problem 5

试用二次量子化方法求出玻色分布 (8 分), 并用幺正变换求出相互作用玻色子系统的超流基态 (5 分).

(1) 对于统计分布下的哈密顿量与粒子数量的算符表示如下:

$$\hat{H}=\sum_l \epsilon_l \hat{a}_l^\dagger \hat{a}_l, \hat{N}=\sum_l \hat{a}_l^\dagger \hat{a}_l. \tag{46}$$

对于量子统计分布的表示为

$$\langle \hat{a}_l^\dagger \hat{a}_l\rangle=\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta\left(\hat{H}-\mu\hat{N}\right)}\hat{a}_l^\dagger \hat{a}_l\right]. \tag{47}$$

定义 $\tilde{\epsilon}_l=\epsilon_l-\mu$, 则:

$$\begin{aligned}\tilde{H}&=\hat{H}-\mu\hat{N}=\sum_l \tilde{\epsilon}_l \hat{a}_l^\dagger \hat{a}_l,\\ Z&=\mathrm{Tr}\left(e^{-\beta\tilde{H}}\right)=\sum_i\langle i|e^{-\beta\tilde{H}}|i\rangle\end{aligned} \tag{48}$$

于是便有

$$\begin{aligned}|i\rangle&=-|n_1,n_2,\ldots,n_m,\ldots\rangle\\ \mathrm{Z}&=\sum_{n_1=0}^{?}\sum_{n_2=0}^{?}\cdots\sum_{n_\infty=0}^{?}e^{-\beta\sum_m\tilde{\epsilon}_m n_m}\\ &=\sum_{n_1=0}^{?}\sum_{n_2=0}^{?}\cdots\sum_{n_\infty=0}^{?}\prod_m e^{-\beta\tilde{\epsilon}_m n_m}\\ &=\prod_m\sum_{n_m=0}^{2}e^{-\beta\tilde{\epsilon}_m n_m},\end{aligned} \tag{49}$$

则对于玻色爱因斯坦分布:

$$\begin{aligned}\langle \hat{a}_l^\dagger \hat{a}_l\rangle&=\frac{\sum_{n_l=0}^{\infty}n_l e^{-\beta\tilde{\epsilon}_l n_l}}{\sum_{n_l=0}^{\infty}e^{-\beta\tilde{\epsilon}_l n_l}}\\ &=\frac{1}{(-\beta)}\frac{\partial}{\partial\tilde{\epsilon}_l}\ln\sum_{n_l=0}^{\infty}e^{-\beta\tilde{\epsilon}_l n_l}\\ &=\frac{1}{(-\beta)}\frac{\partial}{\partial\tilde{\epsilon}_l}\ln\frac{1}{1-e^{-\beta\tilde{\epsilon}_l}}\\ &=\frac{1}{(-\beta)}\frac{(-1)^2(-\beta)e^{-\beta\tilde{\epsilon}_l}}{1-e^{-\beta\tilde{\epsilon}_l}}=\frac{1}{e^{\beta\tilde{\epsilon}_l}-1}\end{aligned} \tag{50}$$

(2) 超流基态满足的本征方程式为:

$$\hat{H}|\mathrm{Bogoliubov}\rangle=E_0|\mathrm{Bogoliubov}\rangle, \tag{51}$$

决定基态的方程式为:

$$\hat{\alpha}_{\vec{\mathbf{p}}\neq 0}|\mathrm{Bogoliubov}\rangle=0, \tag{52}$$

进行幺正变换得：

$$\begin{aligned}\hat{a}_{\vec{\mathbf{p}}} &= \hat{b}_{\vec{\mathbf{p}}} \cosh\theta_{\vec{\mathbf{p}}} + \hat{b}^{\dagger}_{-\vec{\mathbf{p}}} \sinh\theta_{\vec{\mathbf{p}}} = e^{i\hat{S}} \hat{b}_{\vec{\mathbf{p}}} e^{-i\hat{S}}, \\ \hat{\alpha}^{\dagger}_{-\vec{\mathbf{p}}} &= \hat{b}_{\vec{\mathbf{p}}} \sinh\theta_{\vec{\mathbf{p}}} + \hat{b}^{\dagger}_{-\vec{\mathbf{p}}} \cosh\theta_{\vec{\mathbf{p}}} = e^{i\hat{S}} \hat{b}^{\dagger}_{-\vec{\mathbf{p}}} e^{-i\hat{S}}.\end{aligned} \tag{53}$$

只要取：

$$\hat{S} = \frac{i}{2} \sum_{\vec{\mathbf{p}} \neq 0} \theta_{\vec{\mathbf{p}}} (\hat{b}^{\dagger}_{\vec{\mathbf{p}}} \hat{b}^{\dagger}_{-\vec{\mathbf{p}}} - \hat{b}_{-\vec{\mathbf{p}}} \hat{b}_{\vec{\mathbf{p}}}), \tag{54}$$

因此：

$$|\mathrm{Bogoliubov}\rangle = e^{i\hat{S}} |BEC\rangle = \prod_{\vec{\mathbf{p}} \neq 0} \exp\left[\frac{1}{2}\theta_{\vec{\mathbf{p}}} \left(\hat{b}_{-\vec{\mathbf{p}}} \hat{b}_{\vec{\mathbf{p}}} - \hat{b}^{\dagger}_{\vec{\mathbf{p}}} \hat{b}^{\dagger}_{-\vec{\mathbf{p}}}\right)\right] |BEC\rangle. \tag{55}$$