1.3 (p12)

Define a rotation as a matrix that is both orthogonal and special, which satisfies $\boxed{R^T R = I}$ and $\boxed{\det R = 1}$.

The rotation group of the plane consists of the set of all special orthogonal 2-by-2 matrices $SO(2)$.

$$\left.\begin{array}{l} R(\theta) \doteq I + A \\ R^T(\theta) = I + A^T \end{array}\right\} \Rightarrow R^T R = (I + A^T)(I + A) \doteq I + A + A^T = I \Rightarrow A^T = -A$$

A must be antisymmetric.

$\boxed{\text{2-D}}$: $A = \theta J = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$

$$\Rightarrow R = I + \theta J + O(\theta^2) = I + \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} + O(\theta^2) = \begin{pmatrix} 1 & \theta \\ -\theta & 0 \end{pmatrix} + O(\theta^2).$$

The antisymmetric matrix J is known as the generator of the rotation group.

$$R(\theta) = \lim_{N\to\infty} [R(\tfrac{\theta}{N})]^N = \lim_{N\to\infty} [1 + \tfrac{\theta}{N} J]^N = e^{\theta J}$$

$$= \sum_{n=0}^{\infty} \frac{\theta^n J^n}{n!} = \sum_{k=0}^{\infty} \frac{\theta^{2k} J^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{\theta^{2k+1} J^{2k+1}}{(2k+1)!} \qquad (J^2 = -I)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} I + \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} J$$

$$= \cos\theta\, I + \sin\theta\, J$$

$$= \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$\boxed{\text{3-D}}$ Generators are:

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad J_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad J_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and any 3-by-3 antisymmetric matrix can be written as

$$A = \theta_x J_x + \theta_y J_y + \theta_z J_z, \text{ with } \theta_x, \theta_y, \theta_z \in \mathbb{R}. \quad A = \begin{pmatrix} 0 & \theta_z & -\theta_y \\ -\theta_z & 0 & \theta_x \\ \theta_y & -\theta_x & 0 \end{pmatrix}$$

Any 3-d rotation can be written as

$$R(\theta) = e^A = \boxed{e^{\theta_x J_x + \theta_y J_y + \theta_z J_z}} \neq \boxed{e^{\sum_i \theta_i J_i}}, \text{ with } i = x, y, z.$$

To make Js hermitean, define

$$\begin{cases} J_x = -i\, J_x \\ J_y = -i\, J_y \\ J_z = -i\, J_z \end{cases} \Rightarrow \underline{R(\theta) = e^{i \sum_j \theta_j J_j} = e^{i \vec{\theta} \cdot \vec{J}}}$$

$$\left.\begin{aligned} R &\doteq I + A \\ R' &\doteq I + B \end{aligned}\right\} \Rightarrow R R' R^{-1} = (I+A)(I+B)(I-A) \doteq I + B + AB - BA = I + B + [A, B]$$

$$\left.\begin{aligned} A &= i \sum_i \theta_i J_i \\ B &= i \sum_j \theta'_j J_j \end{aligned}\right\} \Rightarrow [A, B] = i \sum_{ij} \theta_i \theta'_j [J_i, J_j]$$

$\downarrow$

The generators of Lie algebra of SO(3)

$[J_i, J_j]$ is itself an antisymmetric 3-by-3 matrix and thus can be written as a linear combination of the J_ks:

$$[J_i, J_j] = i C_{ijk} J_k.$$

$$\left.\begin{aligned} [J_x, J_y] &= i J_z \\ [J_y, J_z] &= i J_x \\ [J_z, J_x] &= i J_y \end{aligned}\right\} \Rightarrow [J_i, J_j] = i \varepsilon_{ijk} J_k$$

IV. 1 (p185)

The elements of $SO(N)$ are represented by the N-by-N matrices transforming the N unit basis vectors $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_N$ into one another. The N-dimensional irreducible representation is furnished by a vector.

Tensor: $T^{ij} \to T'^{ij} = R^{ik} R^{jl} T^{kl}$.

$$N=3. \quad T'^{21} = R^{21}R^{11}T^{11} + R^{21}R^{12}T^{12} + R^{21}R^{13}T^{13}$$
$$+ R^{22}R^{11}T^{21} + R^{22}R^{12}T^{22} + R^{22}R^{13}T^{23}$$
$$+ R^{23}R^{11}T^{31} + R^{23}R^{12}T^{32} + R^{23}R^{13}T^{33}.$$

The tensor T^{ij} consists of 9 objects that transform into linear combinations of themselves under rotations.

- a vector : 1-indexed tensor
- a scalar : 0-indexed tensor

The scalar furnishes the 1-D trivial representation.

$$\underbrace{\begin{pmatrix} D_{11} & \cdots & D_{19} \\ D_{21} & \cdots & D_{29} \\ \vdots & & \vdots \\ D_{91} & \cdots & D_{99} \end{pmatrix}}_{D(R)} \begin{pmatrix} T_{11} \\ \vdots \\ T_{33} \end{pmatrix} = \begin{pmatrix} T'_{11} \\ \vdots \\ T'_{33} \end{pmatrix}$$

$D(R)$ gives a 9-D representation of $SO(3)$

$$T^{ij} \to T'^{ij} = R_1^{ik} R_1^{jl} T^{kl}.$$
$$\Rightarrow T''^{ij} = R_2^{ik} R_2^{jl} T'^{kl} = R_2^{ik} R_1^{km} R_2^{jl} R_1^{ln} T^{mn}$$
$$= (R_2R_1)^{im} (R_2R_1)^{jn} T^{mn}.$$
$$\Rightarrow D(R_2)D(R_1) = D(R_2R_1)$$

or tensor T furnishes the 9-D representation of $SO(3)$.
(reducible)

An antisymmetric tensor $A^{ij} = T^{ij} - T^{ji}$, with the number of independent components is $\frac{1}{2}N(N-1)$.

$N=3$: A^{12}, A^{23}, A^{31}.

$$A^{ij} \to A'^{ij} \equiv T'^{ij} - T'^{ji} = R^{ik}R^{jl}T^{kl} - R^{jk}R^{il}T^{kl} = R^{ik}R^{jl}T^{kl} - R^{jl}R^{ik}T^{lk}$$
$$= R^{ik}R^{jl}(T^{kl} - T^{lk}) = R^{ik}R^{jl}A^{kl}$$

A symmetric tensor $S^{ij} \equiv T^{ij} + T^{ji}$, with the number of indepent components is $\frac{1}{2}N(N-1) + N = \frac{1}{2}N(N+1)$

$$S^{ij} \to S'^{ij} = T'^{ij} + T'^{ji} = R^{ik}R^{jl}T^{kl} + R^{jk}R^{il}T^{kl} = R^{ik}R^{jl}T^{kl} + R^{jl}R^{ik}T^{lk}$$
$$= R^{ik}R^{jl}(T^{kl} + T^{lk}) = R^{ik}R^{jl}S^{kl}.$$

The trace of S, $S^{11}+S^{22}+\cdots+S^{NN}$ transforms into itself:

$$S^{ii} \to S'^{ii} = R^{ik}R^{il}S^{kl} = (R^T)^{ki}R^{il}S^{kl} = (R^{-1})^{ki}R^{il}S^{kl} = \delta^{kl}S^{kl} = S^{kk}.$$

Define a traceless symmetric tensor $\tilde{S}$ for $SO(N)$ by.

$$\tilde{S}^{ij} = S^{ij} - \delta^{ij}\left(\frac{S^{kk}}{N}\right) \xrightarrow{N=3} \tilde{S}^{11}, \tilde{S}^{22}, \tilde{S}^{12}, \tilde{S}^{13}, \tilde{S}^{23}$$

$$S^{-1}D(R)S = \left(\begin{array}{c|c|c} \text{3-by-3 block} & 0 & 0 \\ \hline 0 & \text{1-by-1 block} & 0 \\ \hline 0 & 0 & \text{5-by-5 block} \end{array}\right)$$

$$N^2 - D = \frac{1}{2}N(N-1) + \frac{1}{2}N(N+1) - 1 + 1$$

$N=3$: $9 = 3 \oplus 5 \oplus 1$ 　　　　$N=2$: 4: $1+2+1$

反对称　对称无迹

$N=4$: $16 = 6 \oplus 9 \oplus 1$

$N=5$: $25 = 10 \oplus 14 \oplus 1$

$$R^TR = I \to \delta^{ij}R^{ik}R^{jl} = \delta^{kl} \quad (O)$$

$$\det R = 1 \to \varepsilon^{ijk\cdots n}R^{ip}R^{jq}\cdots R^{ns} = \varepsilon^{pqr\cdots s} \quad (S)$$

$B^{k\cdots n} \to \varepsilon^{ijk\cdots n}R^{ip}R^{jq}A^{pq}$, A and B are said to be dual to each other.

$SO(3)$ 的不可约表示的维数: $\boxed{d = 2j+1}$

$$\frac{1}{2}(j+1)(j+2) - \underbrace{\frac{1}{2}(j-2+1)(j-2+2)}_{\text{无迹条件}} = 2j+1$$

$SO(4)$

the defining or vector representation. $(V^1, V^2, V^3) + V^4$

$4 \to 3 \oplus 1$ spacetime $\to$ space + time

6-D (A^{ij}) antisymmetric　$6 \to 3 \oplus 3$　electromagnetic field $\to$ electric + magnetic fields.

9-D (S^{ij}) symmetric traceless.　$9 \to 5 \oplus 3 + 1$

IV.2 P203.

$$\left.\begin{array}{l}[J_x, J_y]=iJ_z \\ [J_y, J_z]=iJ_x \\ [J_z, J_x]=iJ_y\end{array}\right\} \xrightarrow{J_\pm=J_x\pm iJ_y} [J_z, J_\pm]=\pm J_\pm,\ [J_+,J_-]=2J_z.$$

$$J_zJ_+|m\rangle=(J_+J_z+[J_z,J_+])|m\rangle=(J_+J_z+J_+)|m\rangle=(m+1)J_+|m\rangle$$

$$\Rightarrow J_+|m\rangle=C_{m+1}|m+1\rangle$$

$$J_zJ_-|m\rangle=(J_-J_z+[J_z,J_-])|m\rangle=(J_-J_z-J_-)|m\rangle=(m-1)J_-|m\rangle$$

$$\Rightarrow J_-|m\rangle=b_{m-1}|m-1\rangle$$

$$\langle m+1|J_+|m\rangle=C_{m+1}\Rightarrow C^*_{m+1}=\langle m|(J_+)^\dagger|m+1\rangle=\langle m|J_-|m+1\rangle=\langle m|b_m|m\rangle=b_m$$

$$\Rightarrow b_{m-1}=C^*_m$$

$$\Rightarrow\begin{cases}J_+|m\rangle=C_{m+1}|m+1\rangle\\ J_-|m\rangle=C^*_m|m-1\rangle\end{cases}$$

$$\left.\begin{array}{l}J_+|j\rangle=0\\ [J_+,J_-]=2J_z\end{array}\right\}\Rightarrow 0=\langle j|J_-J_+|j\rangle=\langle j|J_+J_--2J_z|j\rangle=|C_j|^2-2j\Rightarrow|C_j|^2=2j$$

$$\langle m|[J_+,J_-]|m\rangle=\langle m|J_+J_--J_-J_+|m\rangle=|C_m|^2-|C_{m+1}|^2=2\langle m|J_z|m\rangle=2m$$

$$\Rightarrow|C_m|^2=|C_{m+1}|^2+2m$$

从而有

$$J_\pm|m\rangle=\sqrt{j(j+1)-m(m\pm1)}\ |m\pm1\rangle$$

$$|C_{j-1}|^2=|C_j|^2+2(j-1)=2(2j-1)$$

$$|C_{j-2}|^2=|C_{j-1}|^2+2(j-2)=6(j-1)$$

$$\vdots$$

$$|C_{j-s}|^2=2\left((s+1)j-\sum_{i=1}^{s}i\right)=(s+1)(2j-s)$$

s is an integer $\Rightarrow s=2j$ implies that j is an integer or a half-integer.

let $s=j-m$,

$$\Rightarrow|C_m|^2=(j-m+1)(2j-j+m)=(j+m)(j-m+1)$$

$$\Rightarrow J_+|m\rangle=C_{m+1}|m+1\rangle=\sqrt{(j+m+1)(j-m)}\ |m+1\rangle=\sqrt{j(j+1)-m(m+1)}$$

$$J_-|m\rangle = C_m^*|m-1\rangle = \sqrt{(j+1-m)(j+m)}\;|m-1\rangle = \sqrt{j(j+1)-m(m-1)}\;|m-1\rangle$$

Casimir invariant for $SO(3)$: $J^2 = J_x^2 + J_y^2 + J_z^2$

$$J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2$$

$$\Rightarrow J^2|j,m\rangle = \left[\frac{1}{2}(J_+J_- + J_-J_+) + J_z^2\right]|j,m\rangle$$

$$= \left[\frac{1}{2}(|C_m|^2 + |C_{m+1}|^2) + m^2\right]|j,m\rangle$$

$$= \left\{\frac{1}{2}\left[(j+m)(j+1-m) + (j+m+1)(j-m)\right] + m^2\right\}|j,m\rangle$$

$$= (j^2+j)|j,m\rangle$$

$$= j(j+1)|j,m\rangle.$$

IV.3 (p216)

$$l \otimes l' = (l+l') \oplus (l+l'-1) \oplus (l+l'-2) \oplus \cdots \oplus (|l-l'|+1) \oplus |l-l'|$$

$$J_z(|j,m\rangle \otimes |j',m'\rangle) = (m+m')(|j,m\rangle \otimes |j',m'\rangle) = (m+m')|j,j',m,m'\rangle$$

(A) $j = \frac{1}{2}$, $j' = \frac{1}{2}$

$m = -\frac{1}{2}, \frac{1}{2}$, $m' = -\frac{1}{2}, \frac{1}{2}$

$$\overset{J\ M}{|1,1\rangle} = |\uparrow\uparrow\rangle = |\overset{m}{\tfrac{1}{2}}, \overset{m'}{\tfrac{1}{2}}\rangle$$

$$J_-|1,1\rangle = \sqrt{2}|1,0\rangle = J_-|\tfrac{1}{2},\tfrac{1}{2}\rangle = J_-(|\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\rangle) = |-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\rangle + |\tfrac{1}{2}\rangle \otimes |-\tfrac{1}{2}\rangle = |-\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\Rightarrow |1,0\rangle = \frac{1}{\sqrt{2}}\left(|-\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

$$J_-|1,0\rangle = \sqrt{2}|1,-1\rangle = J_-\left[\frac{1}{\sqrt{2}}(|-\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle)\right] = \frac{1}{\sqrt{2}}(|-\tfrac{1}{2},-\tfrac{1}{2}\rangle + |-\tfrac{1}{2},-\tfrac{1}{2}\rangle) = \sqrt{2}|-\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\Rightarrow |1,-1\rangle = |-\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|-\tfrac{1}{2},\tfrac{1}{2}\rangle - |\tfrac{1}{2},-\tfrac{1}{2}\rangle) \qquad \frac{1}{2}\otimes\frac{1}{2} = 1 \oplus 0$$

$$\left.\begin{aligned} |11\rangle &= |\uparrow\uparrow\rangle \\ |10\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1-1\rangle &= |\downarrow\downarrow\rangle \\ |00\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \end{aligned}\right\}$$

(B) $j=1,\ j'=1$

$m, m' = -1, 0, 1 \qquad 3\times3=9$ states.

$$\overset{J\ M}{|2,2\rangle} = \overset{m\ m'}{|1,1\rangle} \quad \checkmark$$

$$|1,1\rangle = |1\rangle \otimes |1\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right) = |2,1\rangle$$

$$J_-|2,1\rangle = \sqrt{6}\,|2,0\rangle = \frac{1}{\sqrt{2}}\left(J_-|1,0\rangle + J_-|0,1\rangle\right) =$$

$$J_-|1,0\rangle = J_-|1\rangle \otimes |0\rangle + |1\rangle \otimes J_-|0\rangle = \sqrt{2}\,|0,0\rangle + \sqrt{2}\,|1,-1\rangle$$

$$J_-|0,1\rangle = J_-|0\rangle \otimes |1\rangle + |0\rangle \otimes J_-|1\rangle = \sqrt{2}\,|-1,1\rangle + \sqrt{2}\,|0,0\rangle$$

$$\Rightarrow \sqrt{6}\,|2,0\rangle = 2|0,0\rangle + |1,-1\rangle + |-1,1\rangle$$

$$\Rightarrow |2,0\rangle = \frac{1}{\sqrt{6}}\left(2|0,0\rangle + |1,-1\rangle + |-1,1\rangle\right)$$

$$|2,-1\rangle = \frac{1}{\sqrt{2}}\left(|0,-1\rangle + |-1,0\rangle\right)$$

$$|2,-2\rangle = |-1,-1\rangle \quad \checkmark \qquad\qquad J=2.$$

$J=1.$ $\quad |0,1\rangle$ & $|1,0\rangle$.

$$\underline{|1,1\rangle = \frac{1}{\sqrt{2}}\left(|0,1\rangle - |1,0\rangle\right)}$$

$$J_-|1,1\rangle = \sqrt{2}\,|1,0\rangle = \frac{1}{\sqrt{2}}\left(J_-|0,1\rangle - J_-|1,0\rangle\right)$$

$$J_-|0,1\rangle = J_-|0\rangle \otimes |1\rangle + |0\rangle \otimes J_-|1\rangle = \sqrt{2}\,|-1,1\rangle + \sqrt{2}\,|0,0\rangle$$

$$J_-|1,0\rangle = J_-|1\rangle \otimes |0\rangle + |1\rangle \otimes J_-|0\rangle = \sqrt{2}\,|0,0\rangle + \sqrt{2}\,|1,-1\rangle$$

$$\Rightarrow \sqrt{2}\,|1,0\rangle = \frac{1}{\sqrt{2}}\sqrt{2}\left(|-1,1\rangle - |1,-1\rangle\right) \Rightarrow \underline{|1,0\rangle = \frac{1}{\sqrt{2}}\left(|-1,1\rangle - |1,-1\rangle\right)}$$

$$\Rightarrow J_-|1,0\rangle = \sqrt{2}\,|1,-1\rangle = \frac{1}{\sqrt{2}}\left(J_-|-1,1\rangle - J_-|1,-1\rangle\right)$$

$$J_-|-1,1\rangle = |-1\rangle \otimes J_-|1\rangle = \sqrt{2}\,|-1\rangle \otimes |0\rangle = \sqrt{2}\,|-1,0\rangle$$

$$J_-|1,-1\rangle = \sqrt{2}\,|0\rangle \otimes |-1\rangle = \sqrt{2}\,|0,-1\rangle$$

$$\Rightarrow \underline{|1,-1\rangle = \frac{\sqrt{2}}{2}\left(|-1,0\rangle - |0,-1\rangle\right).}$$

$$\Rightarrow 1 \otimes 1 = 2 \oplus 1 \oplus 0.$$

$$J=0 \quad |0,0\rangle = \frac{1}{\sqrt{3}}\left(|-1,1\rangle - |0,0\rangle + |1,-1\rangle\right)$$

(c) : $j=2, j'=1$. ~~$m, m' = -3, -2, -1, 0, 1, 2, 3$~~

$m = -2, -1, 0, 1, 2$. $m' = -1, 0, 1$

$|3,3\rangle = |2,1\rangle$

$J_-|3,3\rangle = \sqrt{6}|3,2\rangle = J_-|2\rangle \otimes |1\rangle + |2\rangle \otimes J_-|1\rangle = 2|1,1\rangle + \sqrt{2}|2,0\rangle$

$J_-|2\rangle = \sqrt{6-2}\,|1\rangle = 2|1\rangle \quad J_-|1\rangle = \sqrt{2}|0\rangle$

$\Rightarrow |3,2\rangle = \frac{2}{\sqrt{6}}|1,1\rangle + \frac{1}{\sqrt{3}}|2,0\rangle$ ✓

$\Rightarrow |3,1\rangle = \frac{1}{\sqrt{15}}(\sqrt{6}|0,1\rangle + 2\sqrt{2}|1,0\rangle + |2,-1\rangle)$

⋮

$|3,-3\rangle$

$\Rightarrow 2 \otimes 1 = 3 \oplus 2 \oplus 1$

$m + m' = 2$.

d $j = 1, j' = \frac{1}{2}$ $m = -1, 0, 1$ $m' = -\frac{1}{2}, \frac{1}{2}$

$|\frac{3}{2}, \frac{3}{2}\rangle = |1, \frac{1}{2}\rangle$ (1)

$J_-|\frac{3}{2}, \frac{3}{2}\rangle = \sqrt{3}|\frac{3}{2}, \frac{1}{2}\rangle = J_-|1\rangle \otimes |\frac{1}{2}\rangle + |1\rangle \otimes J_-|\frac{1}{2}\rangle = \sqrt{2}|0, \frac{1}{2}\rangle + |1, -\frac{1}{2}\rangle$

$\Rightarrow |\frac{3}{2}, \frac{1}{2}\rangle = \frac{\sqrt{2}}{\sqrt{3}}|0, \frac{1}{2}\rangle + \frac{1}{\sqrt{3}}|1, -\frac{1}{2}\rangle$ (2)

$J_-|\frac{3}{2}, \frac{1}{2}\rangle = 2|\frac{3}{2}, -\frac{1}{2}\rangle = \frac{\sqrt{2}}{\sqrt{3}} J_-|0, \frac{1}{2}\rangle + \frac{1}{\sqrt{3}} J_-|1, -\frac{1}{2}\rangle$ -

$J_-|0, \frac{1}{2}\rangle = J_-|0\rangle \otimes |\frac{1}{2}\rangle + |0\rangle \otimes J_-|\frac{1}{2}\rangle = \frac{\sqrt{15}}{2}|-1, \frac{1}{2}\rangle + |0, -\frac{1}{2}\rangle$ ✓

$J_-|1, -\frac{1}{2}\rangle = J_-|1\rangle \otimes |-\frac{1}{2}\rangle + |1\rangle \otimes J_-|-\frac{1}{2}\rangle = \frac{\sqrt{15}}{2}|0, -\frac{1}{2}\rangle +$

$\Rightarrow |\frac{3}{2}, -\frac{1}{2}\rangle = \cancel{\frac{1}{\sqrt{6}}(\frac{\sqrt{15}}{2}|-1, \frac{1}{2}\rangle + |0, -\frac{1}{2}\rangle) + \frac{1}{2\sqrt{3}}\frac{\sqrt{7}}{2}}|0, -\frac{1}{2}\rangle$

$= \cancel{\frac{\sqrt{5}}{2\sqrt{2}}|-1, \frac{1}{2}\rangle} + \frac{1}{\sqrt{6}}|0, -\frac{1}{2}\rangle$

$\frac{\sqrt{2}}{2\sqrt{3}}(\frac{\sqrt{15}}{2}|-1, \frac{1}{2}\rangle + |0, -\frac{1}{2}\rangle) + \frac{1}{\sqrt{3}}\frac{\sqrt{15}}{2}|0, -\frac{1}{2}\rangle$

$= \frac{\sqrt{10}}{4}|-1, \frac{1}{2}\rangle + \frac{\sqrt{2}}{2\sqrt{3}}|0, -\frac{1}{2}\rangle + \frac{\sqrt{5}}{2}|0, -\frac{1}{2}\rangle$

2v.4 p227

$U(N)$: consist of all N by N matrices U that are unitary

$U^\dagger U = I$

prove. $(U_2U_1)^\dagger U_2 U_1 = U_1^\dagger U_2^\dagger U_2 U_1 = U_1^\dagger (U_2^\dagger U_2) U_1 = U_1^\dagger U_1 = I$.

$U^\dagger U = I \xrightarrow{\text{determinant}} \det(U^\dagger U) = (\det U^\dagger)(\det U) = (\det U)^*(\det U) = |\det U|^2 = 1$

$\Rightarrow \det U = e^{i\alpha} = 1$

$U(1)$: consisting of all 1-by-1 unitary matrices.
or all complex numbers with absolute value equal to 1
$\Rightarrow$ $U(1)$ consists of phase factors $e^{i\varphi}$, $0 \le \varphi < 2\pi$.

$U(N)$ $\begin{cases} \text{unitary matrices of the form } e^{i\varphi} I,\ I: N\text{-by-}N \text{ identity matrix } (U(1)) \\ N\text{-by-}N \text{ unitary matrices with determinant equal to } 1\ (SU(N)) \end{cases}$

The dimension of the representation furnished by the 3-indexed total symmetric tensor ψ^{ijk} of $SO(3)$.

$\psi^{333}, \psi^{332}, \psi^{331}, \psi^{311}, \psi^{321}, \psi^{322}, \psi^{222}, \psi^{221}, \psi^{211}, \psi^{111}$: 10

~~SO(3)~~ SU(3) Lie algebra.

$U \simeq I + iH$, H is arbitary "small" complex matrix.

$\Rightarrow U^\dagger U \simeq (1 - iH^\dagger)(1 + iH) \simeq I - i(H^\dagger - H) = I \Rightarrow H^\dagger = H$, means H is hermitean.

$\Rightarrow \boxed{U = e^{iH}}$; U is unitary when H is hermitean.

H can always be diagonalized and write $H = W^\dagger \Lambda W$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$
W is a unitary matrix.

$\det U = \det e^{iH} = \det e^{iW^\dagger \Lambda W} = \det(W^\dagger e^{i\Lambda} W) = \det(WW^\dagger) \det e^{i\Lambda}$

$= \det e^{i\Lambda} = \prod_{j=1}^{N} e^{i\lambda_j} = e^{i\sum_{j=1}^{N}\lambda_j} = e^{i\,\mathrm{tr}\Lambda} = e^{i\,\mathrm{tr}\, W\Lambda W} = e^{i\,\mathrm{tr}H} = 1$

$\Rightarrow \mathrm{tr}H = 0$ N-by-N traceless hermitean matrix H.

$N=2$.

$$\begin{pmatrix} u & w \\ z & v \end{pmatrix}^{\dagger} = \begin{pmatrix} u & w \\ z & v \end{pmatrix} \Rightarrow H = \frac{1}{2}\begin{pmatrix} \theta_3 & \theta_1 - i\theta_2 \\ \theta_1 + i\theta_2 & -\theta_3 \end{pmatrix} , \quad \theta_1, \theta_2, \theta_3 \in \mathbb{R}$$

Define 3 traceless hermitean matrices, known as Pauli matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow$ The most general 2-by-2 traceless hermitean matrix H can be written as: $\boxed{H} = \frac{1}{2}(\theta_1\sigma_1 + \theta_2\sigma_2 + \theta_3\sigma_3) = \boxed{\sum_{a=1}^{3} \frac{1}{2}\theta_a\sigma_a}$

An element of $SU(2)$ can be written as $\boxed{U = e^{i\theta_a\sigma_a/2}}$

$N=3$. Define the eight 3-by-3 traceless hermitean matrices.

Gell-Mann matrices.

$$H = \theta_a \frac{\lambda_a}{2}, \quad U = e^{i\theta_a \frac{\lambda_a}{2}} , \quad a = 1, 2, \cdots, 8$$

The number of real numbers (θ_a) required to specify a general N-by-N traceless hermitean matrix is given by $N-1+2\cdot\frac{1}{2}N(N-1) = \boxed{N^2-1}$

$U = e^{i\theta^a T^a}$, $\theta^a \in \mathbb{R}$, T^a's are generators of $SU(N)$

$$\left.\begin{array}{l} U_1 \simeq I + A \\ U_2 \simeq I + B \end{array}\right\} \Rightarrow U_2^{-1}U_1U_2 = (I-B)(I+A)(I+B) \simeq I + A + AB - BA = I + A + [A, B]$$

antihermitean and traceless.

$$\left.\begin{array}{l} A = i\sum_a \theta^a T^a \\ B = i\sum_b \theta'^b T^b \end{array}\right\} \Rightarrow [A, B] = i^2 \sum_{ab} \theta^a \theta'^b [T^a, T^b]$$

$\Rightarrow [T^a, T^b] = i f^{abc} T^c$

f^{abc} for $SU(2)$ and $SU(3)$ are.

SU(N): the group formed by unitary matrices with unit determinant.

IV.5 p244.

(1) SU(2) locally isomorphic to SO(3) ☆
(2) SU(2) covers SO(3) twice. ☆

$$X = x\sigma_1 + y\sigma_2 + z\sigma_3 = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$ hermitean and traceless.

and its determinant is $\det X = -\vec{x}^2 = -(x^2+y^2+z^2)$

$X' = U^+ X U$ is hermitean and traceless.

$(X'^+) = (U^+XU)^+ = U^+X^+U = U^+XU = X'$

$\operatorname{tr} X' = \operatorname{tr} U^+XU = \operatorname{tr} X U U^+ = \operatorname{tr} X = 0.$

$\Rightarrow X' = \vec{x}' \cdot \vec{\sigma}$, $\vec{x}' = (x', y', z')$

$\det X' = -(\vec{x}')^2 = \det U^+XU = (\det U^+)(\det X)(\det U) = \det X = -\vec{x}^2$

$\Rightarrow$ $\vec{x}'$ and $\vec{x}$ are linearly related to each other and also have the same length.
means that the 3-vector $\vec{x}$ is rotated into 3-vector $\vec{x}'$
defines a rotation R

$\Rightarrow$ We can associate an element R of SO(3) with any element U of SU(2).

$U \to R$ defines group multiplication

$U_1 \to R_1$, $U_2 \to R_2$ then $U_1U_2 \to R_1R_2$

$(U_1U_2)^+ X (U_1U_2) = U_2^+(U_1^+XU_1)U_2 = U_2^+ X' U_2 = X''$

$$\begin{array}{ccccc} \vec{x} & \to & \vec{x}' & \to & \vec{x}'' \\ & \downarrow & & \downarrow & \\ & R_1 & & R_2 & \end{array}$$

the map $f: U \to R$ of SU(2) into SO(3) is 2-to-1. since U and $-U$ are mapped into the same R. $f(U) = f(-U)$

Since $(-U)^+X(-U) = U^+XU$. U and $-U$ are manifestly not the same.

The unitary group SU(2) is said to double cover the orthogonal group SO(3).

SU(2) and SO(3) are only locally isomorphic.

Properties of the Pauli matrices.

$(\sigma_a)^2 = I, \quad a = 1, 2, 3.$

~~$\sigma_1\sigma_2 = i\sigma_3$~~

$$\sigma_a \sigma_b = \delta_{ab} I + i \varepsilon_{abc} \sigma_c \Rightarrow \{\sigma_a, \sigma_b\} = 2\delta_{ab}$$
$$[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$$

$\Rightarrow \left[\frac{\sigma_a}{2}, \frac{\sigma_b}{2}\right] = i\varepsilon_{abc}\frac{\sigma_c}{2}$

$[T^a, T^b] = i f^{abc} T^c$

$\Rightarrow$ For the fundamental 2-D representation, the generators T^a are represented by $\frac{\sigma_a}{2}$

the structure constant of SU(2) is ε^{abc}

The Lie algebra of SU(2) is given by $[T^a, T^b] = i\varepsilon^{abc}T^c$ or

☆

$$SU(2)\begin{cases}[T^1, T^2] = iT^3\\ [T^2, T^3] = iT^1\\ [T^3, T^1] = iT^2\end{cases} \qquad SO(3)\begin{cases}[J_x, J_y] = iJ_z\\ [J_y, J_z] = iJ_x\\ [J_z, J_x] = iJ_y\end{cases}$$

The two Lie algebras are manifestly identical, isomorphic.

Lie algebra of SU(2)

$$\begin{cases}[T^3, T^\pm] = \pm T^\pm\\ [T^+, T^-] = 2T^3\end{cases}$$

the representations of SU(2) are $\boxed{2j+1}$ D

$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$

2-D representation with $j = \frac{1}{2}$ ($|-\frac{1}{2}\rangle$ and $|\frac{1}{2}\rangle$) is the fundamental or defining representation of the SU(2) Group.

$$(\vec{u}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma}) = (\vec{u}\cdot\vec{v})I + i(\vec{u}\otimes\vec{v})\cdot\vec{\sigma}$$

$$U = e^{i\vec{\varphi}\cdot\vec{\sigma}/2} = \sum_{n=0}^{\infty}\frac{i^n}{n!}\left(\frac{\vec{\varphi}\cdot\vec{\sigma}}{2}\right)^n = \left\{\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{\varphi}{2}\right)^{2k}\right\}I + i\left\{\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{\varphi}{2}\right)^{2k+1}\right\}\hat{\varphi}\cdot\vec{\sigma}$$

$$= \cos\frac{\varphi}{2}I + i\hat{\varphi}\cdot\vec{\sigma}\sin\frac{\varphi}{2}$$

$$U = e^{i\varphi\sigma_3/2} = \begin{pmatrix} e^{i\frac{\varphi}{2}} & 0\\ 0 & e^{-i\frac{\varphi}{2}}\end{pmatrix} = \begin{pmatrix}\cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2} & 0\\ 0 & \cos\frac{\varphi}{2} - i\sin\frac{\varphi}{2}\end{pmatrix} = \cos\frac{\varphi}{2}I + i\sigma_3\sin\frac{\varphi}{2}$$

Take $\hat{\varphi}$ to point along the third axis. For $a=3$, $U^{+}\sigma_3 U = \sigma_3$

For $a=1,2$,

$$U^{+}\sigma_a U = \left(\cos\frac{\varphi}{2} I - i\sigma_3 \sin\frac{\varphi}{2}\right)\sigma_a\left(\cos\frac{\varphi}{2} I + i\sigma_3\sin\frac{\varphi}{2}\right)$$

$$= \left(\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}\right)\sigma_a - i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}[\sigma_3, \sigma_a]$$

$$\Rightarrow \quad U^{+}\sigma_1 U = \cos\varphi\,\sigma_1 + \sin\varphi\,\sigma_2$$

$$U^{+}\sigma_2 U = -\sin\varphi\,\sigma_1 + \cos\varphi\,\sigma_2$$

$$X' = U^{+}(x\sigma_1 + y\sigma_2 + z\sigma_3)U = (\cos\varphi\, x - \sin\varphi\, y)\sigma_1 + (\sin\varphi\, x + \cos\varphi\, y)\sigma_2 + z\sigma_3$$

$$\Rightarrow \begin{cases} x' = \cos\varphi\, x - \sin\varphi\, y \\ y' = \sin\varphi\, x + \cos\varphi\, y \\ z' = z \end{cases}$$

$\rightarrow$ for a rotation around the z-axis through angle φ.

$$U(\varphi) = e^{i\varphi\sigma_3/2} \Rightarrow U(2\pi) = e^{i2\pi\sigma_3/2} = e^{i\pi\sigma_3} = \begin{pmatrix} e^{i\pi} & 0 \\ 0 & e^{-i\pi} \end{pmatrix} = -I$$

φ goes from 0 to 2π, the corresponding rotation has gone back to the identity

the SU(2) element U has reached only $-I$

$\Rightarrow$ $U(4\pi) = I$ the SU(2) double covers SO(3).

By the time we get around SU(2) once, the corresponding rotation has gone around SO(3) twice.

The 2-D fundamental representation $\psi^i \to U^i_j \psi^j$ of SU(2) is strictly speaking not a representation of SO(3). but a double-valued representation of SO(3)

SU(2) and SO(3) are not globally isomorphic, only locally isomorphic, only their algebras are isomorphic.

Dimension of SU(2) irreducible representations $2j+1$.

an irreducible representation of SU(2) is characterized by j. which can take on integral or half-integral values.

$j=\frac{1}{2}$: fundamental representation.

$j=1$: the vector representation.

247, (p261).

The character of rotation.

$$\left.\begin{array}{l} J_3|jm\rangle = m|jm\rangle \\ e^{i\psi J_3}|jm\rangle = e^{im\psi}|jm\rangle \end{array}\right\} \Rightarrow \boxed{\chi(j,\psi) = \frac{\sin(j+\frac{1}{2})\psi}{\sin\frac{\psi}{2}}}$$

Proof. $\chi(j,\psi) = \sum_{m=-j}^{j} e^{im\psi} = e^{ij\psi} + e^{i(j-1)\psi} + \cdots + e^{-ij\psi}$

$$= e^{-ij\psi}\left(e^{2ij\psi} + e^{i(2j-1)\psi} + \cdots + 1\right) = e^{-ij\psi}\frac{e^{i(2j+1)\psi}-1}{e^{i\psi}-1} = \frac{\sin(j+\frac{1}{2})\psi}{\sin\frac{\psi}{2}}$$

$\Rightarrow \chi(j,0) = 2j+1$

Integration measure

$$\int_{SO(3)} d\mu(g)\, \chi(k,\psi)^* \chi(j,\psi) = \int_0^{\pi} d\psi \sin^2\left(\frac{\psi}{2}\right) \frac{\sin(k+\frac{1}{2})\psi \sin(j+\frac{1}{2})\psi}{\sin^2\frac{\psi}{2}}$$

$$= \int_0^{\pi} d\psi \sin(k+\tfrac{1}{2})\psi \sin(j+\tfrac{1}{2})\psi$$

$$= \frac{1}{2}\int_0^{\pi} d\psi \left(\cos(j-k)\psi - \cos(j+k+1)\psi\right) = \boxed{\frac{\pi}{2}\delta_{jk}.}$$

for $j \neq k$. ~~by~~ ~~ch~~ the characters of the irreducible representations j and k are orthogonal to each other.